ON THE FUNCTORIALITY OF KHOVANOV-FLOER THEORIES

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Abstract. We introduce the notion of a *Khovanov-Floer theory*. Roughly, such a theory assigns a filtered chain complex over $\mathbb{Z}/2\mathbb{Z}$ to a link diagram such that (1) the E_2 page of the resulting spectral sequence is naturally isomorphic to the Khovanov homology of the link; (2) this filtered complex behaves nicely under planar isotopy, disjoint union, and 1-handle addition; and (3) the spectral sequence collapses at the E_2 page for any diagram of the unlink. We prove that a Khovanov-Floer theory naturally yields a functor from the link cobordism category to the category of spectral sequences. In particular, every page (after E_1) of the spectral sequence accompanying a Khovanov-Floer theory is a link invariant, and an oriented cobordism in $\mathbb{R}^3 \times [0,1]$ between links in $\mathbb{R}^3$ induces a map between each page of their spectral sequences, invariant up to smooth isotopy of the cobordism rel boundary.

We then show that the spectral sequences relating Khovanov homology to Heegaard Floer homology and singular instanton knot homology are induced by Khovanov-Floer theories and are therefore *functorial* in the manner described above, as has been conjectured for some time. We further show that Szabó's *geometric spectral sequence* comes from a Khovanov-Floer theory, and is thus functorial as well. In addition, we illustrate how our framework can be used to give another proof that Lee's spectral sequence is functorial and that Rasmussen's invariant is a knot invariant. Finally, we use this machinery to define some potentially new knot invariants.

1. Introduction

Khovanov's groundbreaking paper [20] associates to a link diagram a bigraded chain complex whose homology is, up to isomorphism, an invariant of the underlying link type. This invariant *categorifies* the Jones polynomial in the sense that the graded Euler characteristic of Khovanov homology is equal to the Jones polynomial. One reason to promote a polynomial-valued invariant to a group-valued invariant is that it makes sense to talk about morphisms between groups; groups form a category. This extra structure is often useful. In the case of Khovanov homology with $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients, Jacobsson showed [18] that a *movie* for a cobordism in $\mathbb{R}^3 \times [0,1]$ with starting and ending diagrams D_0 and D_1 induces a map

$$Kh(D_0) \to Kh(D_1),$$

and that *equivalent* movies define the same map (see also [4, 21, 10]). In other words, Khovanov homology is really a functor

$$Kh : \mathbf{Diag} \to \mathbf{Vect}_{\mathbb{F}}$$

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from the diagrammatic link cobordism category (see Subsection 2.3) to the category of vector spaces over $\mathbb{F}$.

Rasmussen put this additional structure to spectacular use in [29], combining this functoriality with work of Lee [24] to define a numerical invariant of knots which provides a lower bound on the smooth 4-ball genus. He then used this invariant to compute the smooth 4-ball genera of torus knots, affirming a conjecture of Milnor first proven by Kronheimer and Mrowka using gauge theory [22]. When combined with work of Freedman, Quinn, and Rudolph [12, 33], Rasmussen's proof of Milnor's conjecture also provides the first existence result for exotic $\mathbb{R}^4$'s which avoids gauge theory, Floer homology, or any significant tools from analysis.

Categorification has also played a major role in establishing connections between quantum invariants and Floer homology. These now ubiquitous connections generally take the form of a spectral sequence having Khovanov homology as its E_2 page and converging to the relevant Floer-homological invariant. The first such connection was discovered by Ozsváth and Szabó in [28]. Given a based link $L \subset \mathbb{R}^3$ with diagram D, they defined a spectral sequence with E_2 page the reduced Khovanov homology of D, converging to the Heegaard Floer homology $\widehat{HF}(-\Sigma(L))$ of the branched double cover of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ along L with reversed orientation. Similar spectral sequences in monopole, framed instanton, and plane Floer homology have since been discovered by Bloom, Scaduto, and Daemi, respectively [6, 8, 1]. Perhaps most significantly, Kronheimer and Mrowka defined in [23] a spectral sequence with E_2 page the Khovanov homology of D, converging to the singular instanton knot homology $I^\sharp(\overline{L})$ of the mirror of L. This spectral sequence played a central role in their celebrated proof that Khovanov homology detects the unknot [23]. In addition to their structural significance, these and related spectral sequence have been used to:

- study the knot Floer homology of fibered knots [31, 30],
- establish tightness and non-fillability of certain contact structures [3],
- prove that Khovanov homology detects the unknot [23],
- prove that Khovanov's categorification of the n-colored Jones polynomial detects the unknot for $n \geq 2$ [13],
- detect the unknot with Khovanov homology of certain satellites [15, 17],
- prove that Khovanov homology detects the unlink [16, 5].
- relate Khovanov homology to the twist coefficient of braids [14].

Each of these spectral sequences arises in the standard way (via exact couples) from a filtered chain complex associated with a link diagram and some additional, often analytic, data. However, one can generally show that the (E_i, d_i) page of the resulting spectral sequence does not depend on this additional data, up to canonical isomorphism, for $i \geq 2$. Indeed, we may think of Kronheimer and Mrowka's construction as assigning to a planar diagram D for a link L a sequence

$$KM(D) = \{(E_i^{KM}(D), d_i^{KM}(D))\}_{i\geq 2}$$

with

$$E_2^{KM}(D) = Kh(D) \text{ and } E_\infty^{KM}(D) \cong I^\sharp(\overline{L}).$$

Likewise, Ozsváth and Szabó's construction assigns to a planar diagram D for a based link L a sequence

$$OS(D) = \{(E_i^{OS}(D), d_i^{OS}(D))\}_{i\geq 2}$$

with

$$E_2^{OS}(D) = Khr(D) \quad \text{and} \quad E_\infty^{OS}(D) \cong \widehat{HF}(-\Sigma(L)).$$

Given that the E_2 and E_∞ pages of these spectral sequence are, up to isomorphism, link type invariants, a natural question is whether all intermediate pages are as well. Affirmative answers to this question were given in [2] and [23] for the Heegaard Floer and singular instanton Floer spectral sequences, respectively. In this paper, we consider natural extensions of this question, regarding both the invariance *and* the functoriality of these spectral sequences and of their relatives. That is, of all spectral sequences given by what we call *Khovanov-Floer theories*.

For now, let us continue the discussion of functoriality in the instanton and Heegaard Floer cases. We write **Link** to denote the link cobordism category, whose objects are oriented links in $\mathbb{R}^3$, and whose morphisms are isotopy classes of oriented, collared link cobordisms in $\mathbb{R}^3 \times [0,1]$. In particular, two surfaces represent the same morphism if they differ by smooth isotopy fixing a collar neighborhood of the boundary pointwise. As explained in Subsection 2.3, Khovanov homology can be made into a functor

$$Kh : \mathbf{Link} \to \mathbf{Vect}_{\mathbb{F}}$$

in a natural way. Meanwhile, Kronheimer and Mrowka showed that a cobordism S from L_0 to L_1 gives rise to a map on singular instanton knot homology,

$$I^\sharp(-S) : I^\sharp(\overline{L}_0) \to I^\sharp(\overline{L}_1),$$

which is an invariant of the morphism in **Link** represented by S. That is, singular instanton knot homology also defines a functor

$$I^\sharp : \mathbf{Link} \to \mathbf{Vect}_{\mathbb{F}}.$$

So, in essence, the E_2 and E_∞ pages of Kronheimer and Mrowka's spectral sequence behave functorially with respect to link cobordism. It is therefore natural to ask, as Kronheimer and Mrowka did, whether their *entire* spectral sequence (after the E_1 page) defines a functor from **Link** to the spectral sequence category $\mathbf{Spect}_{\mathbb{F}}$, of which an object is a sequence $\{(E_i, d_i)\}_{i \geq i_0}$ of chain complexes over $\mathbb{F}$ satisfying

$$H_*(E_i, d_i) = E_{i+1},$$

and a morphism is a sequence of chain maps

$$\{F_i : (E_i, d_i) \to (E_i', d_i')\}_{i \geq i_0}$$

satisfying $F_{i+1} = (F_i)_*$. We record their question informally as follows.

Question 1.1. (Kronheimer-Mrowka [23, Section 8.1]) *Is the spectral sequence from Khovanov homology to singular instanton knot homology functorial?*

One can ask a similar question of Ozsváth and Szabó's spectral sequence. For any $p \in \mathbb{R}^3$, reduced Khovanov homology can be thought of as a functor

$$Khr : \mathbf{Link}_p \to \mathbf{Vect}_{\mathbb{F}},$$

where $\mathbf{Link}_p$ is the based link cobordism category, whose objects are oriented links in $\mathbb{R}^3$ containing p, and whose morphisms are isotopy classes of oriented, collared link cobordisms in $\mathbb{R}^3 \times [0,1]$ containing the arc $\{p\} \times [0,1]$. In this category, two surfaces represent the same morphism if they differ by smooth isotopy fixing both a collar neighborhood of the boundary and this arc pointwise. Given a based link cobordism S from L_0 to L_1, the branched double cover of $S^3 \times [0,1]$ along S is a smooth, oriented 4-dimensional cobordism $\Sigma(S)$ from $\Sigma(L_0)$ to $\Sigma(L_1)$, and therefore induces a map on Heegaard Floer homology

$$\widehat{HF}(-\Sigma(S)) : \widehat{HF}(-\Sigma(L_0)) \to \widehat{HF}(-\Sigma(L_1)),[1]$$

which is an invariant of the morphism in $\mathbf{Link}_p$ represented by S. That is, the Heegaard Floer homology of branched double covers defines a functor

$$\widehat{HF}(\Sigma(\cdot)) : \mathbf{Link}_p \to \mathbf{Vect}_{\mathbb{F}}$$

as well. This leads to the natural question, posed informally by Ozsváth and Szabó, as to whether their spectral sequence defines a functor from $\mathbf{Link}_p$ to $\mathbf{Spect}_{\mathbb{F}}$.

Question 1.2. (Ozsváth-Szabó [28, Section 1.1]) *Is the spectral sequence from Khovanov homology to the Heegaard Floer homology of the branched double cover functorial?*

In this paper, we answer both Questions 1.1 and 1.2 in the affirmative. Indeed, we prove that Kronheimer-Mrowka's and Ozsváth-Szabó's spectral sequences are functorial, expressed more precisely in the two theorems below. In these theorems,

$$SV_j : \mathbf{Spect}_{\mathbb{F}} \to \mathbf{Vect}_{\mathbb{F}}$$

is the forgetful functor which sends $\{(E_i, d_i)\}_{i \geq i_0}$ to its jth page E_j.

Theorem 1.3. *There exists a functor*

$$KM : \mathbf{Link} \to \mathbf{Spect}_{\mathbb{F}}$$

with $\mathrm{Kh} = SV_2 \circ KM$ *such that* $KM(L) \cong KM(D)$ *for any diagram* D *for* L.

Theorem 1.4. *There exists a functor*

$$OS : \mathbf{Link}_p \to \mathbf{Spect}_{\mathbb{F}}$$

with $\mathrm{Khr} = SV_2 \circ OS$ *such that* $OS(L) \cong OS(D)$ *for any diagram* D *for* L.

In particular, proper isotopy classes of link cobordisms induce well-defined maps on the intermediate pages of these spectral sequences, which agree at E_2 with the induced maps on Khovanov and reduced Khovanov homology.

One notable consequence of these theorems is that link isotopies determine isomorphisms of these spectral sequences. In particular, an isotopy ϕ taking L to L' determines a cylindrical cobordism $S_\phi \subset \mathbb{R}^3 \times [0,1]$ from L to L', and, therefore, a morphism

$$\Psi_\phi := KM(S_\phi) : KM(L) \to KM(L')$$

(likewise for based isotopies and OS). Since this cobordism is an isomorphism in $\mathbf{Link}$, the morphism Ψ_ϕ is an isomorphism in $\mathbf{Spect}_{\mathbb{F}}$. In this way, we recover the results from [2] and [23] that the isomorphism classes of the intermediate pages of these spectral sequences are link type invariants.

Theorems 1.3 and 1.4 follow from a much more general framework developed in this paper. The key idea is the notion of a *Khovanov-Floer theory*, alluded to above and formally introduced in Section 3. Very roughly, this is something which assigns a filtered chain complex to a link diagram (and possibly extra data) such that (1) the E_2 page of the resulting spectral sequence is naturally isomorphic to the Khovanov homology of the diagram; (2) the filtered complex behaves in

[1]This map is usually denoted by $F_{-\Sigma(S)}$.

certain nice ways under planar isotopy, disjoint union, and diagrammatic 1-handle addition; and (3) the spectral sequence collapses at the E_2 page for any diagram of the unlink. The import of this notion is indicated by our main theorem below, which asserts that the spectral sequence associated with a Khovanov-Floer theory is automatically functorial.

Theorem 1.5. *The spectral sequence associated with a Khovanov-Floer theory defines a functor*

$$F : \mathbf{Link} \to \mathbf{Spect}_{\mathbb{F}}$$

with $\mathrm{Kh} = SV_2 \circ F$.

In particular, the spectral sequence defined by a Khovanov-Floer theory is, up to isomorphism, a link type invariant. What is striking is how this invariance and the additional functoriality promised in the theorem are guaranteed by just the few, rather weak conditions that go into the definition of a Khovanov-Floer theory.

To prove Theorem 1.5, we first show that the spectral sequence associated with a Khovanov-Floer theory defines a functor from **Diag** to $\mathbf{Spect}_{\mathbb{F}}$. The morphism of spectral sequences this functor assigns to a movie is induced by a filtered chain map between the filtered complexes associated with the diagrams at either end of the movie. To define this filtered chain map, we represent the movie as a composition of *elementary* movies, each corresponding to a planar isotopy, diagrammatic handle attachment, or Reidemeister move. We assign a filtered map to each elementary movie so that the induced map on E_2 agrees with the corresponding Khovanov map, and we define the map associated with the original movie to be the composite of these elementary movie maps. For planar isotopy and handle attachment, these elementary maps are essentially built into the definition of a Khovanov-Floer theory. More interesting is our assignment of filtered maps to Reidemeister moves. The idea is to first arrange via movie moves that the Reidemeister move takes place amongst unknotted components. Then one constructs the desired map using the behavior of a Khovanov-Floer theory under disjoint union and handle attachment, and the fact that the associated spectral sequence collapses at E_2 for any diagram of an unlink. The fact that equivalent movies are assigned equal morphisms (so that we actually get a functor from **Diag**) follows immediately from the fact that these morphisms agree on E_2 with the corresponding Khovanov map. Finally, we promote this to a functor from **Link** in a relatively standard way.

The power of our framework lies in the fact it is often easy to determine whether a given construction satisfies the conditions of a Khovanov-Floer theory. In Section 5, we show that several well-known constructions are indeed Khovanov-Floer theories. Importantly, we prove the following.

Theorem 1.6. *Kronheimer-Mrowka's and Ozsváth-Szabó's spectral sequences come from Khovanov-Floer theories.*[2]

Observe that Theorems 1.3 and 1.4 follow immediately from Theorem 1.6 combined with Theorem 1.5. Though we do not do so here, one can show that the spectral sequences defined by Bloom, Scaduto, and Daemi also come from Khovanov-Floer theories and are therefore functorial as well.

[2]Really, Ozsváth and Szabó's construction is what we term a *reduced* Khovanov-Floer theory.

The other examples in Section 5 concern constructions which do *not* come from Floer homology. The first of these is Szabó's *geometric spectral sequence* [34], which relates the Khovanov homology of L to another combinatorial link type invariant which (though defined without Floer homology) is conjecturally isomorphic to

$$\widehat{HF}(-\Sigma(L)) \oplus \widehat{HF}(-\Sigma(L)).$$

We prove the following.

Theorem 1.7. *Szabó's spectral sequence comes from a Khovanov-Floer theory.*

Theorem 1.7 provides an alternative proof of Szabó's result that this spectral sequence is a link type invariant, while furthermore showing that it behaves functorially with respect to link cobordism.

For another example, we consider Lee's deformation of Khovanov homology [24]. For knots, this deformation produces a spectral sequence converging to the direct sum $\mathbb{F}\oplus\mathbb{F}$, with each summand supported in a single quantum grading. Rasmussen's invariant, mentioned earlier, may be described as the average of these two gradings. We can easily prove the following.

Theorem 1.8. *Lee's spectral sequence comes from a Khovanov-Floer theory.*[3]

This yields an alternative proof that Lee's spectral sequence is a link type invariant, from which it follows that Rasmussen's invariant $s_{\mathbb{F}}$ is as well.

Apart from their theoretical appeal, we expect our functoriality results to have applications to computing Floer theories and the maps on Floer homology induced by link cobordisms. Indeed, in the singular instanton and Heegaard Floer settings, one can show that the morphism of spectral sequences we assign to a cobordism is induced by a filtered chain map whose induced map on total homology agrees with the cobordism map on Floer homology. In the case of Kronheimer and Mrowka's construction, for example, this means that there is a commutative diagram

$$\begin{tikzcd} H_*(C(D_0)) \arrow[r, "(f_M)_*"] \arrow[d, "\cong"'] & H_*(C(D_1)) \arrow[d, "\cong"] \\ I^\sharp(\overline{L}_0) \arrow[r, "I^\sharp(-S)"'] & I^\sharp(\overline{L}_1). \end{tikzcd}$$

Here, $C(D_i)$ is the filtered complex associated to a diagram D_i for a link L_i which gives rise to Kronheimer and Mrowka's spectral sequence, and f_M is the filtered chain map associated to a movie M for the cobordism S which induces the morphism of spectral sequences

$$KM(S) : KM(L_0) \to KM(L_1)$$

in Theorem 1.3. The third author and Zentner [26] recently used the idea that diagrammatic 1-handle additions induce morphisms of spectral sequences to compute the singular instanton and Heegaard Floer spectral sequences for a variety of knots, even without the assumption (proved in this paper) that the morphism associated

[3]We actually prove this for a version of Lee's spectral sequence defined over $\mathbb{F}$ by Bar-Natan [4].

to a movie for a cobordism is independent of the movie. The functoriality established here should allow us to extend these sorts of calculations to a wider array of knots.

The notion of a Khovanov-Floer theory can also be helpful in proving topological invariance for Floer-homological constructions when establishing invariance by more conventional means is challenging. For example, Herald, Kirk and the second author recently defined a Lagrangian Floer analogue of singular instanton knot homology, which they call *pillowcase Floer homology*. A direct proof of the topological invariance of their theory is quite difficult, but they should be able to bypass this difficulty by showing that pillowcase Floer homology is isomorphic to the E_∞ page of the spectral sequence associated with a Khovanov-Floer theory.

In a slightly different direction, the results in this paper imply that any reasonably well-behaved deformation of the Khovanov chain complex gives rise to link and cobordism invariants. This suggests a mechanism for constructing a wealth of new invariants. To illustrate this principle, we construct in Subsection 5.5 some new deformations of the Khovanov complex which are easily shown to define Khovanov-Floer theories. At the moment, however, we do not know whether the resulting link and cobordism invariants are any different from those in Khovanov homology. A natural (and probably very difficult) problem is to classify the link invariants that come from Khovanov-Floer theories.

1.1. Organization.

In Section 2, we collect some facts from homological algebra and review Khovanov homology and ideas involving functoriality. In Section 3, we give a precise definition of a Khovanov-Floer theory. In Section 4, we prove our main result, Theorem 1.5. In Section 5, we show that the spectral sequence constructions of Kronheimer-Mrowka, Ozsváth-Szabó, Szabó, and Lee constitute Khovanov-Floer theories, and we describe some new deformations of the Khovanov complex which also define Khovanov-Floer theories.

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2. Background

We will work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout the entire paper unless otherwise specified.

2.1. Homological algebra.

In this subsection, we record some basic results about filtered chain complexes and their associated spectral sequences.

The filtered chain complexes considered in this paper are all chain complexes over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, admitting a direct sum decomposition of the form

$$(C = \bigoplus_{i \geq i_0} C^i,\, d = d^0 + d^1 + \dots), \tag{1}$$

where:

- $d^i(C^j) \subset C^{j+i}$ for each $j \geq i_0$, and
- $C^i = \{0\}$ for all i greater than some i_1.

We consider elements of C^i to be homogeneous of *grading* i. This grading should not be confused with a (co)homological grading (i.e. a grading raised by one by d)

which, while generally present, will be suppressed throughout the discussion. The associated filtration

$$C = \mathscr{F}^{i_0} \supset \mathscr{F}^{i_0+1} \supset \cdots \supset \mathscr{F}^{i_1} = \{0\} \tag{2}$$

is given by

$$\mathscr{F}^i = \bigoplus_{j \geq i} C^j.$$

In fact, *every* filtered complex over $\mathbb{F}$ (or any other field) can be thought of in terms of a graded complex in which the differential does not decrease grading, as above. From this perspective, a *filtered chain map of degree k* from (C, d) to (C', d') is a chain map $f : C \to C'$ admitting a splitting

$$f = f^k + f^{k+1} + f^{k+2} + \dots \tag{3}$$

such that $f^i(C^j) \subset (C')^{j+i}$.

A *spectral sequence* is a sequence of chain complexes $\{(E_i, d_i)\}_{i \geq i_0}$ for some $i_0 \geq 0$ satisfying

$$E_{i+1} = H_*(E_i, d_i).$$

A filtered complex (C, d) gives rise to a spectral sequence

$$\{(E_i(C), d_i(C))\}_{i \geq 0}$$

of graded vector spaces via the standard *exact couple* construction; see, e.g. [7, Section 14]. Note that each $E_i(C)$ inherits a grading from that of C. As usual, we will write $E_i(C) = E_\infty(C)$ to mean that

$$E_i(C) = E_{i+1}(C) = E_{i+2}(C) = \cdots := E_\infty(C).$$

A *morphism* from a spectral sequence $\{(E_i, d_i)\}_{i \geq i_0}$ to a spectral sequence $\{(E'_i, d'_i)\}_{i \geq i'_0}$ is a sequence of chain maps

$$\{F_i : (E_i, d_i) \to (E'_i, d'_i)\}_{i \geq \max\{i_0, i'_0\}}$$

satisfying $F_{i+1} = (F_i)_*$. A filtered chain map as in (3) gives rise to a morphism of spectral sequences

$$\{F_i = E_i(f) : (E_i(C), d_i(C)) \to (E_i(C'), d_i(C'))\}_{i \geq 0}$$

in a standard way as well. If the filtered map is of degree k, then each map in the morphism is homogenous of degree k with respect to the grading. As mentioned in the introduction, spectral sequences and their morphisms form a category which we denote by $\mathbf{Spect}_{\mathbb{F}}$.

The three lemmas below are the main results of this subsection; we will make heavy use of them in Sections 3 and 4.

Lemma 2.1. *Suppose*

$$f : (C, d) \to (C', d')$$

is a degree 0 *filtered chain map such that $E_i(f)$ is an isomorphism. Then $E_j(f)$ is an isomorphism for all $j \geq i$. Moreover, there exists a degree* 0 *filtered chain map*

$$g : (C', d') \to (C, d)$$

such that $E_j(g) = E_j(f)^{-1}$ for all $j \geq i$.

Lemma 2.2. *Suppose*

$$f, g : (C, d) \to (C', d')$$

are degree k filtered chain maps such that $E_i(f) = E_i(g)$. Then $E_j(f) = E_j(g)$ for all $j \geq i$.

Lemma 2.3. *Suppose $E_i(C) = E_\infty(C)$. Then there exists a degree 0 filtered chain map*

$$f : (C, d) \to (E_i(C), 0)$$

from (C, d) to the complex consisting of the vector space $E_i(C)$ with trivial differential such that the induced map

$$E_i(f) : E_i(C) \to E_i(C)$$

is the identity map.

The remainder of this section is devoted to proving these lemmas (even though they are well-known to experts). We will do so using a procedure called *cancellation* which provides a concrete way of understanding these spectral sequences and the maps between them. We first describe this procedure for ordinary (unfiltered) chain complexes, as part of the well-known *cancellation lemma* below.

Lemma 2.4 (Cancellation Lemma)**.** *Suppose (C, d) is a chain complex over $\mathbb{F}$ freely generated by elements $\{x_i\}$ and let $d(x_i, x_j)$ be the coefficient of x_j in $d(x_i)$. If $d(x_k, x_l) = 1$, then the complex (C', d') with generators $\{x_i | i \neq k, l\}$ and differential*

$$d'(x_i) = d(x_i) + d(x_i, x_l)d(x_k)$$

is chain homotopy equivalent to (C, d) via the chain homotopy equivalences

$$\pi : C \to C' \quad \textit{and} \quad \iota : C' \to C$$

given by

$$\pi = P \circ (id + d \circ h) \quad \textit{and} \quad \iota = (id + h \circ d) \circ I,$$

where P and I are the natural projection and inclusion maps and h is the linear map defined by

$$h(x_l) = x_k \quad \textit{and} \quad h(x_i) = 0 \textit{ for } i \neq l.$$

We say that the complex (C', d') is obtained from (C, d) by canceling *the component of d from x_k to x_l.*

Remark 2.5. The homology $H_*(C, d)$ of the complex in Lemma 2.4 can be understood as the vector space obtained by performing cancellation until the resulting differential is zero. Technically, the actual vector space resulting from this cancellation depends on the order of cancellations, but any such vector space is canonically isomorphic to $H_*(C, d)$.

Suppose now that (C, d) is a filtered chain complex as in (1). One may think of the sequence $\{E_i(C)\}_{i \geq 0}$ as the sequence of graded vector spaces obtained by performing cancellation in stages, where the ith *page* records the result of this cancellation after the ith stage. Specifically, let:

- $(C_{(0)}, d_{(0)}) = (C, d)$, and inductively let
- $(C_{(i)}, d_{(i)})$ be the complex obtained from $(C_{(i-1)}, d_{(i-1)})$ by canceling the components of $d_{(i-1)}$ which shift the grading by $i - 1$.

Then $E_i(C)$ may be thought of as the graded vector space $C_{(i)}$, with grading naturally inherited from C. Under this formulation, the spectral sequence differential $d_k(C)$ on $E_k(C)$ is the sum of the components of $d_{(k)}$ which shift the grading by exactly k, so that the recursive condition above may be interpreted as the more familiar

$$E_i(C) = H_*(E_{i-1}(C), d_{i-1}(C)),$$

per Remark 2.5.

Suppose that f is a filtered chain map of degree k as in (3). Cancellation provides a nice way of understanding the induced maps

$$E_i(f) : E_i(C) \to E_i(C')$$

for each $i \geq 0$. Specifically, every time we cancel a component of d or d', we may adjust the components of f as though they were components of a differential (they *are* components of the mapping cone differential). In this way, we obtain an *adjusted map*

$$f_{(i)} : (C_{(i)}, d_{(i)}) \to (C'_{(i)}, d'_{(i)})$$

for each $i \geq 0$. The induced map $E_i(f)$ may then be understood as the sum of the components of $f_{(i)}$ which shift the grading by exactly k. Note that if

$$f : (C, d) \to (C', d') \quad \text{and} \quad g : (C', d') \to (C'', d'')$$

are filtered chain maps of degrees j and k, respectively, then $g \circ f$ is naturally a degree $j + k$ filtered chain map, and

$$E_i(g \circ f) = E_i(g) \circ E_i(f)$$

for all $i \geq 0$.

Remark 2.6. A degree k filtered chain map f can also be thought of as a degree j map for any $j \leq k$. On the other hand, the definition of $E_i(f)$ depends on the degree of f. It is therefore important that one specifies the degree of f when talking about these induced maps.

Remark 2.7. Given a degree k filtered chain map f from (C, d) to (C', d'), it is worth pointing out that

$$E_\infty(f) : E_\infty(C) \to E_\infty(C')$$

does not necessarily agree with the the induced map

$$f_* : H_*(C, d) \to H_*(C', d'),$$

via the isomorphisms between the domains and codomains. In fact, it can be the case that f_* is an isomorphism while $E_\infty(f)$ is the zero map e.g. regard the identity map as a degree -1 filtered chain map. What is true, however, is that

$$f_* = E_\infty(f) + \text{higher order terms}$$

where "higher order terms" means terms in the decomposition of the adjusted map $f_{(\infty)} = f_*$ according to the grading that shift the grading by more than k.

Remark 2.8. Note that for each cancellation performed in computing the spectral sequence associated to a filtered complex (C, d), the maps π and ι of Lemma 2.4 are degree 0 filtered chain maps. In particular, by taking compositions of these maps, we obtain degree 0 filtered chain maps

$$\pi_{(i)} : (C, d) \to (C_{(i)}, d_{(i)}) \quad \text{and} \quad \iota_{(i)} : (C_{(i)}, d_{(i)}) \to (C, d)$$

for each $i \geq 0$. Tautologically, we have that the induced maps

$$E_j(\pi_{(i)}) : E_j(C) \to [E_j(C_{(i)}) = E_j(C)]$$
$$E_j(\iota_{(i)}) : [E_j(C_{(i)}) = E_j(C)] \to E_j(C)$$

are the identity maps for all $j \geq i$.

Below, we prove Lemmas 2.1, 2.2, and 2.3 using the above descriptions of spectral sequences and induced maps in terms of cancellation.

Proof of Lemma 2.1. Suppose f is a map as in the lemma and let

$$f_{(i)} : (C_{(i)}, d_{(i)}) \to (C'_{(i)}, d'_{(i)})$$

be the adjusted map as defined above. The fact that $E_i(f)$ is an isomorphism implies that $f_{(i)}$ is too. Moreover, it is easy to see that its inverse

$$g_{(i)} = f^{-1}_{(i)} : (C'_{(i)}, d'_{(i)}) \to (C_{(i)}, d_{(i)})$$

is also a filtered chain map of degree 0, and that $E_j(f_{(i)})$ and $E_j(g_{(i)})$ are inverses for all $j \geq i$. Let

$$g : (C', d') \to (C, d)$$

be the degree 0 filtered chain map given by $g = \iota_{(i)} \circ g_{(i)} \circ \pi_{(i)}$ for maps

$$\pi_{(i)} : (C', d') \to (C'_{(i)}, d'_{(i)}) \quad \text{and} \quad \iota_{(i)} : (C_{(i)}, d_{(i)}) \to (C, d)$$

as in Remark 2.8. Then $E_j(f) = E_j(f_{(i)})$ and $E_j(g) = E_j(g_{(i)})$ are inverses for all $j \geq i$. In particular, each $E_j(f)$ is an isomorphism. □

Proof of Lemma 2.2. It is clear from the discussion above that if a filtered chain map induces the zero map on some page then it induces the zero map on all subsequent pages. Now suppose $E_i(f) = E_i(g)$ as in the lemma. Then

$$E_i(f - g) = E_i(f) - E_i(g) = 0,$$

which implies that

$$E_j(f) - E_j(g) = E_j(f - g) = 0$$

for all $j \geq i$, completing the proof. □

Proof of Lemma 2.3. Note that $(E_i(C), 0) = (C_{(i)}, d_{(i)})$ in this case. We may therefore take f to be the map

$$f = \pi_{(i)} : (C, d) \to (C_{(i)}, d_{(i)}),$$

per Remark 2.8. □

2.2. **Khovanov homology.** In this subsection, we review the definitions and some basic properties of Khovanov homology and its reduced variant.

Suppose D is a planar diagram for an oriented link in $\mathbb{R}^3$, with crossings labeled $1, \dots, n$. Let n_+ and n_- denote the numbers of positive and negative crossings of D. For each $I \in \{0,1\}^n$, let I_j denote the jth coordinate of I and let D_I be the diagram obtained by taking the I_j-resolution (as shown in Figure 1) of the jth crossing of D, for every $j \in \{1, \dots, n\}$. Let $V(D_I)$ be the vector space generated by the components of D_I. We endow $\Lambda^* V(D_I)$ with a grading $\mathbf{p}$ according to the rules that $1 \in \Lambda^0 V(D_I)$ has grading $\mathbf{p}(1) = m$, where m is equal to the number of components of D_I, and that wedging with any of the components decreases the $\mathbf{p}$ grading by 2.

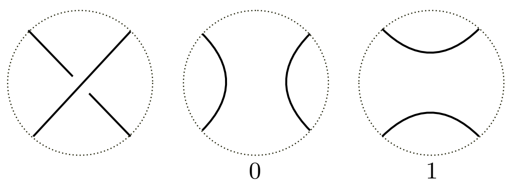

FIGURE 1. The 0- and 1-resolutions of a crossing.

Given tuples $I, J \in \{0,1\}^n$, we write $I <_k J$ if J may be obtained from I by changing exactly k 0s to k 1s. For each pair I, I' with $I <_1 I'$, one defines a map

$$d_{I,I'} : \Lambda^* V(D_I) \to \Lambda^* V(D_{I'}),$$

as described below. The Khovanov chain complex assigned to D is then given by

$$CKh(D) = \bigoplus_{I \in \{0,1\}^n} \Lambda^* V(D_I),$$

with differential

$$d = \bigoplus_{I <_1 I'} d_{I,I'}.$$

This is a bigraded complex, with (co-)*homological grading* defined by

$$\mathbf{h}(x) = I_1 + \cdots + I_n - n_-,$$

for $x \in \Lambda^* V(D_I)$, and *quantum grading* defined by

$$\mathbf{q}(x) = \mathbf{p}(x) + \mathbf{h}(x) + n_+ - n_-,$$

for homogeneous $x \in \Lambda^* V(D_I)$. The differential d increases $\mathbf{h}$ by one and preserves $\mathbf{q}$. Thus, if we write $CKh^{i,j}(D)$ for the summand of $CKh(D)$ in homological grading i and quantum grading j, then d restricts to a differential on

$$CKh^{*,j}(D) = \bigoplus_i CKh^{i,j}(D)$$

for each j. We will write

$$Kh^{i,j}(D) = H_i(CKh^{*,j}(D), d)$$

for the (co-)homology of this complex in homological grading i. The *Khovanov homology of D* refers to the bigraded vector space

$$Kh(D) = \bigoplus_{i,j} Kh^{i,j}(D).$$

Remark 2.9. We will also treat the case in which D is the *empty* diagram. In this case, we let $Kh(D) = CKh(D) = \Lambda^*(0) = \mathbb{F}$.

It remains to define $d_{I,I'}$. Note that the diagram $D_{I'}$ is obtained from D_I either by merging two circles into one or by splitting one circle into two. Suppose first that $D_{I'}$ is obtained by merging the components x and y of D_I into one circle. Then there is an obvious identification

$$V(D_{I'}) \cong V(D_I)/(x+y),$$

and we define the *merge* map $d_{I,I'}$ to be the induced quotient map

$$\Lambda^* V(D_I) \to \Lambda^*(V(D_I)/(x+y)) \cong \Lambda^* V(D_{I'}).$$

Suppose next that $D_{I'}$ is obtained by splitting a component of D_I into two circles x and y. Then the identification

$$V(D_I) \cong V(D_{I'})/(x+y)$$

induces an identification

$$\Lambda^* V(D_I) \cong \Lambda^*(V(D_{I'})/(x+y)) \cong (x+y) \wedge \Lambda^* V(D_{I'}),$$

and we define the *split* map $d_{I,I'}$ to be the composition of the maps

$$\Lambda^* V(D_I) \xrightarrow{\cong} \Lambda^*(V(D_{I'})/(x+y)) \xrightarrow{\cong} (x+y) \wedge \Lambda^* V(D_{I'}) \xrightarrow{\subset} \Lambda^* V(D_{I'}).$$

That is, the split map may be thought of as given by wedging with $x+y$.

For diagrams D and D' which differ by a Reidemeister move, Khovanov defines in [20] an isomorphism

$$Kh(D) \to Kh(D'),$$

which we refer to as the *standard* isomorphism associated to the Reidemeister move. In this way, the isomorphism class of Khovanov homology provides an invariant of oriented link type.

Next, we describe how the theory behaves under disjoint union. Consider the link diagram $D \sqcup D'$ obtained as a disjoint union of diagrams D and D'. Suppose D has m crossings and D' has n crossings. For $I \in \{0,1\}^m$ and $I' \in \{0,1\}^n$, let $II' \in \{0,1\}^{m+n}$ denote the tuple formed via concatenation. Note that for every such II', there is a canonical isomorphism

$$V((D \sqcup D')_{II'}) \to V(D_I) \oplus V(D_{I'}),$$

which naturally induces an isomorphism

$$\Lambda^* V((D \sqcup D')_{II'}) \to \Lambda^* V(D_I) \otimes \Lambda^* V(D_{I'}).$$

The direct sum of these isomorphisms define an isomorphism

$$CKh(D \sqcup D) \to CKh(D) \otimes CKh(D'),$$

that induces an isomorphism

$$Kh(D \sqcup D) \to Kh(D) \otimes Kh(D'),$$

which we refer to as the *standard* isomorphism associated to disjoint union.

In *reduced* Khovanov homology, one considers planar diagrams with a basepoint on one component (in particular, all such diagrams are nonempty). Suppose D is a planar diagram with basepoint p. Consider the chain map

$$\Phi_p : CKh(D) \to CKh(D)$$

given on each $V(D_I)$ by wedging with the component of D_I with the basepoint. The image of this map is a subcomplex of $CKh(D)$. The reduced Khovanov complex of D is defined to be the associated quotient complex,

$$CKhr(D) := (CKh(D)/\mathrm{Im}(\Phi_p))[0,-1]. \tag{4}$$

The reduced Khovanov homology

$$Khr(D) = H_*(CKhr(D))$$

is then the bigraded vector space obtained as the homology of this quotient complex. In (4), the bracketed term $[0,-1]$ indicates a shift of the (i,j) bigrading by $(0,-1)$. This shift is introduced so that the reduced Khovanov homology of the unknot is supported in bigrading $(0,0)$.

In reduced Khovanov homology, Reidemeister moves away from the basepoint give rise to isomorphisms of Khovanov groups. In particular, the isomorphism class of reduced Khovanov homology provides an invariant of *based*, oriented link type.

Reduced Khovanov homology behaves under connected sum similar to the way Khovanov homology behaves under disjoint union. Suppose D and D' are disjoint diagrams based at points p and p' that are adjacent to some common region in the complement of these diagrams. Let $D\#D'$ be a connected sum of these diagrams, where the connected sum band is attached in this common region, along the edges containing these basepoints, so that p and p' are on the same edge of this connected sum diagram. Choose p or p' as the basepoint for $D\#D'$. Then there is a natural isomorphism

$$Khr(D\#D') \to Khr(D) \otimes Khr(D').$$

Remark 2.10. Note that there is a natural isomorphism between $Khr(D \sqcup U)$ and $Kh(D)$, where U is the crossingless diagram of the unknot containing the basepoint.

2.3. **Functoriality.** In this subsection, we review some categorical aspects of links, cobordisms, and their diagrams. We then describe how Khovanov homology defines a functor from various cobordism categories to $\mathbf{Vect}_{\mathbb{F}}$.

The category we will be most interested in is the *link cobordism category* **Link**. Objects of **Link** are oriented links in $\mathbb{R}^3$ and morphisms are proper isotopy classes of collared, smoothly embedded link cobordisms in $\mathbb{R}^3 \times [0,1]$. This means that two surfaces represent the same morphism if they differ by smooth isotopy fixing a neighborhood of the boundary pointwise. In order to define a functor from **Link**, one often starts by defining a functor from the *diagrammatic link cobordism category* **Diag** mentioned in the introduction. This category can be thought of as a more combinatorial model for **Link**. We define this category below and then describe how functors from **Diag** can be turned into functors from **Link**, focusing on the case of Khovanov homology.

Objects of **Diag** are oriented link diagrams in $\mathbb{R}^2$ and morphisms are *movies* up to *equivalence*. We define these two terms below. A *movie* is a 1-parameter family D_t, $t \in [0,1]$, where the D_t are link diagrams except at finitely many t-values where the topology of the diagram changes by a local move consisting of a Reidemeister move or a Morse modification (a diagrammatic handle attachment). Away from these exceptional t-values, the link diagrams vary by planar isotopy. Movies M_1 and M_2 can be composed in a natural way $M_2 \circ M_1$, assuming that the initial diagram of M_2 agrees with the terminal diagram of M_1. Then any movie can be described as a finite composition of *elementary movies*, where each elementary movie corresponds to either:

- a Reidemeister move (of type I, II, or III), or
- an oriented diagrammatic handle attachment (a 0-, 1-, or 2-handle), or
- a planar isotopy of diagrams.

Carter and Saito [9] refer to the first two types of elementary movies as *elementary string interactions* (ESIs). We will generally represent an ESI diagrammatically by recording diagrams just before and just after the corresponding change in topology. Figure 2 shows the ESIs corresponding to handle attachments.

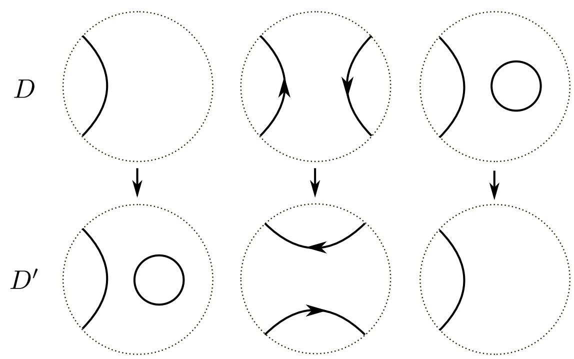

FIGURE 2. From left to right, oriented diagrammatic 0-, 1-, and 2-handle attachments.

Note that a movie M defines an immersed surface $\Sigma_M \subset \mathbb{R}^2 \times [0,1]$ with

$$D_t = \Sigma_M \cap (\mathbb{R}^2 \times \{t\}).$$

We refer to these cross sections as the *levels* of Σ_M. We will often think of a movie *as* its corresponding immersed surface and vice versa. Let

$$\pi : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$$

be projection on the first two factors. Given links $L_0, L_1 \subset \mathbb{R}^3$ with $\pi(L_i) = D_i$, we can lift Σ_M to a link cobordism $S \subset \mathbb{R}^3 \times [0,1]$ from L_0 to L_1 such that

$$(\pi \times id)(S) = \Sigma_M.$$

As **Diag** is supposed to serve as a model for **Link**, we ought to declare two movies from D_0 to D_1 to be equivalent if their lifts, for fixed L_0 and L_1, represent the same morphism in **Link**. Carter and Saito discovered how to interpret this equivalence diagrammatically in [9]. In particular, two movies are *equivalent* if they can be related by a finite sequence of the following moves:

- the *movie moves* of Carter and Saito [9, Figures 23-37],
- level-preserving isotopies (of their associated immersed surfaces),
- interchange of the levels containing *distant* ESIs.

We will not describe these moves in detail as we do not need them; we refer to the reader to [9] for more information.

Khovanov homology, as described in the previous subsection, assigns a vector space to a link diagram. To extend Khovanov homology to a functor from **Diag** to

$\mathbf{Vect}_{\mathbb{F}}$, one must assign maps to movies such that equivalent movies are assigned the same map. We describe below how this is done, following Jacobsson [18].

First, one assigns maps to elementary movies. To an elementary movie M from D_0 to D_1 corresponding to a Reidemeister move, we assign the associated *standard* isomorphism

$$Kh(M) : Kh(D_0) \to Kh(D_1)$$

mentioned in the previous subsection. Suppose M is the movie corresponding to a planar isotopy ϕ taking D_0 to D_1. This isotopy determines a canonical isomorphism

$$F_\phi : CKh(D_0) \to CKh(D_1).$$

We assign to M the induced map on homology,

$$Kh(M) := (F_\phi)_* : Kh(D_0) \to Kh(D_1).$$

It remains to assign a map to a movie M from D_0 to D_1 corresponding to an oriented i-handle attachment, for $i = 0, 1, 2$.

For $i = 0$, the diagram D_1 is a disjoint union $D_0 \sqcup U$, where U is the crossingless diagram of the unknot. It follows that

$$Kh(D_1) \cong Kh(D_0) \otimes Kh(U) = Kh(D_0) \otimes \Lambda^*(\mathbb{F}\langle U\rangle),$$

and we define

$$Kh(M) : Kh(D_0) \to Kh(D_0) \otimes \Lambda^*(\mathbb{F}\langle U\rangle)$$

to be the map which sends x to $x \otimes 1$ for all $x \in Kh(D_0)$.

Similarly, for $i = 2$, we can view D_0 as a disjoint union $D_1 \sqcup U$, so that

$$Kh(D_0) \cong Kh(D_1) \otimes \Lambda^*(\mathbb{F}\langle U\rangle).$$

In this case, we define

$$Kh(M) : Kh(D_1) \otimes \Lambda^*(\mathbb{F}\langle U\rangle) \to Kh(D_0)$$

to be the map which sends $x \otimes 1$ to 0 and $x \otimes U$ to x for all $x \in Kh(D_1)$.

Finally, for $i = 1$, each complete resolution $(D_1)_I$ is obtained from $(D_0)_I$ via a merge or split. The merge and split maps used to define the differential on Khovanov homology therefore give rise to a map

$$\Lambda^* V((D_0)_I) \to \Lambda^* V((D_1)_I).$$

These maps fit together to define a chain map

$$CKh(D_0) \to CKh(D_1),$$

and $Kh(M)$ is the induced map on homology. Put slightly differently, let $\tilde{D}$ be a diagram with one more crossing than D_0 and D_1 such that D_0 is the 0-resolution of $\tilde{D}$ at this crossing c and D_1 is the 1-resolution (we will think of c as the $(n+1)^{\text{st}}$ crossing). Then the Khovanov complex for $\tilde{D}$ is the mapping cone of the chain map

$$T : CKh(D_1) \to CKh(D_1),$$

given by the direct sum

$$T = \bigoplus_{I \in \{0,1\}^n} d_{I\times\{0\}, I\times\{1\}},$$

where these

$$d_{I\times\{0\}, I\times\{1\}} : \Lambda^* V((D_0)_I) \to \Lambda^* V((D_1)_I)$$

are components of the differential on $CKh(\tilde{D})$. Then

$$Kh(M) := T_* : Kh(D_0) \to Kh(D_1).$$

Given an arbitrary movie M from D_0 to D_1, expressed as a composition

$$M = M_1 \circ \cdots \circ M_k$$

of elementary movies, we then define

$$Kh(M) : Kh(D_0) \to Kh(D_1)$$

to be the composition

$$Kh(M) = Kh(M_k) \circ \cdots \circ Kh(M_1).$$

In this way, Khovanov homology assigns maps to movies. The key theorem is the following result from [18]; see also [4, 21].

Theorem 2.11. (Jacobsson [18]) *If M and M' are equivalent movies, then* $\mathrm{Kh}(M) = \mathrm{Kh}(M')$.

Jacobsson proves this theorem by showing that the maps assigned to movies are invariant under the moves listed above. As desired, his result implies that Khovanov homology defines a functor

$$Kh : \mathbf{Diag} \to \mathbf{Vect}_{\mathbb{F}}.$$

We next consider how to lift this and other functors from **Diag** to functors from **Link**. We shall achieve this by defining functors,

$$\Pi_\alpha : \mathbf{Link} \to \mathbf{Diag}.$$

To define Π_α, we take for every link $L \subset \mathbb{R}^3$ a choice of smooth isotopy ϕ_L^α which begins at L and ends at a link $\phi_L^\alpha(L)$ on which the projection

$$\pi : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$$

restricts to a regular immersion. We will also regard such an isotopy as a morphism

$$\phi_L^\alpha \in \mathrm{Mor}(L, \phi_L^\alpha(L)),$$

represented by the smoothly embedded cylinder obtained from its trace. On objects, we define Π_α by

$$\Pi_\alpha(L) := \pi(\phi_L^\alpha(L)).$$

Given a morphism $S \in \mathrm{Mor}(L_0, L_1)$, let us consider the associated morphism

$$\phi_{L_1}^\alpha \circ S \circ (\phi_{L_0}^\alpha)^{-1} \in \mathrm{Mor}(\phi_{L_0}^\alpha(L_0), \phi_{L_1}^\alpha(L_1)).$$

According to [9, Theorem 5.2, Remark 5.2.1(2)], there is a representative Σ of this morphism whose image under the projection

$$\pi \times id : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^2 \times [0,1]$$

is a movie. We define $\Pi_\alpha(S)$ to be the equivalence class of this movie,

$$\Pi_\alpha(S) := [(\pi \times id)(\Sigma)].$$

Proposition 2.12. $\Pi_\alpha : \mathbf{Link} \to \mathbf{Diag}$ *is a functor.*

Proof. Clearly Π_α is well-defined on objects. To see that it is well-defined on morphisms, we use the relative version of Carter and Saito's main result [9, Theorem 7.1], which states that properly isotopic surfaces project to equivalent movies. Thus, the movies resulting from the projections of any two representatives of the morphism $\phi^\alpha_{L_1} \circ S \circ (\phi^\alpha_{L_0})^{-1}$ are equivalent. □

The apparent dependence of the functor Π_α on the choices of isotopies ϕ^α_L is undesirable. In fact, we have the following.

Proposition 2.13. *Suppose that* $\{\phi^\alpha_L\}$ *and* $\{\phi^\beta_L\}$ *are two collections of isotopies to links with regular projections, as above, defining functors*

$$\Pi_\alpha, \Pi_\beta : \mathbf{Link} \to \mathbf{Diag}.$$

Then the functors Π_α *and* Π_β *are naturally isomorphic.*

Proof. The assignment θ^β_α which sends a link L to the morphism

$$\theta^\beta_\alpha(L) := [(\pi \times id)(\Sigma)] \in \mathrm{Mor}(\Pi_\alpha(L), \Pi_\beta(L)),$$

where Σ is a representative of the morphism $\phi^\beta_L \circ (\phi^\alpha_L)^{-1}$ whose image under $\pi \times id$ is a movie, gives a well-defined natural isomorphism from Π_α to Π_β. Commutativity of the square

$$\begin{tikzcd}
\Pi_\alpha(L_0) \arrow[r, "\Pi_\alpha(S)"] \arrow[d, "\theta^\beta_\alpha(L_0)"'] & \Pi_\alpha(L_1) \arrow[d, "\theta^\beta_\alpha(L_1)"] \\
\Pi_\beta(L_0) \arrow[r, "\Pi_\beta(S)"'] & \Pi_\beta(L_1)
\end{tikzcd}$$

follows from the work of Carter and Saito; we leave it as an exercise. It is also not hard to show that θ^α_β is the inverse natural transformation, and that

$$\theta^\gamma_\beta \circ \theta^\beta_\alpha = \theta^\gamma_\alpha$$

for any three collections of isotopies. □

Moreover we have

Proposition 2.14. *For any choice of isotopies* ϕ^α_L*, the functor* $\Pi_\alpha : \mathbf{Link} \to \mathbf{Diag}$ *is an equivalence of categories.*

Proof. Since any two such functors are naturally isomorphic, it is enough to verify the proposition for a good choice of isotopies ϕ^α_L. We take isotopies ϕ^α_L such that if L is already regularly immersed under the map π, then ϕ^α_L is the identity isotopy. Hence we have that Π_α is surjective on objects. Furthermore, Π_α is bijective on morphism sets (that is, it is full and faithful), which suffices to establish the equivalence by, e.g. [27, Theorem 1, IV.4]. Surjectivity on morphisms is easy since movies can easily be lifted to cobordisms in $\mathbb{R}^3 \times [0,1]$, whereas injectivity on morphisms is again a consequence of [9, Theorem 7.1]. □

One can then lift Khovanov homology to a functor from **Link** by precomposing with any Π_α. We shall denote this functor by

$$Kh_\alpha := Kh \circ \Pi_\alpha : \mathbf{Link} \to \mathbf{Vect}_{\mathbb{F}}.$$

This functor assigns vector spaces to links, but these vector spaces depend on extra data, the extra data being the set of isotopies $\{\phi_L^\alpha\}_{L\subset\mathbb{R}^3}$ to links with regular projections. We would prefer a functor which assigns vector spaces to links themselves, and does not depend on the choice of isotopies. Our solution rests on the natural isomorphisms we have described between the functors Π_α.

Indeed, using notation from the proof of Proposition 2.13, we obtain isomorphisms

$$Kh_\alpha^\beta(L) := Kh(\theta_\alpha^\beta(L)) : Kh_\alpha(L) \to Kh_\beta(L)$$

satisfying $Kh_\beta^\gamma \circ Kh_\alpha^\beta = Kh_\alpha^\gamma$ and $Kh_\alpha^\alpha = \mathrm{Id}$, for all α, β, γ. Thus the collection of vector spaces $\{Kh_\alpha(L)\}_\alpha$ and isomorphisms $\{Kh_\alpha^\beta(L)\}_{\alpha,\beta}$ form a transitive system in the sense of [11, Chapter 1.6]. We define $Kh(L)$ to be the vector space associated to this system in the standard way, as the inverse limit over the complete directed graph on the set of isotopies. A morphism $S \in \mathrm{Mor}(L_0, L_1)$ then gives rise to a well-defined map

$$Kh(S) : Kh(L_0) \to Kh(L_1),$$

so that Kh defines a functor from **Link** to $\mathbf{Vect}_{\mathbb{F}}$ which is independent of any choice of isotopies, as desired.

Remark 2.15. Of course, one should not expect that Kh can be lifted to a functor associating a vector space to each *isotopy class* of link. This is because there exist knots with self-isotopies inducing non-identity automorphisms of the Khovanov invariant.

We conclude this section by noting that for every $p \in \mathbb{R}^3$, reduced Khovanov homology defines a similar functor

$$Khr : \mathbf{Link}_p \to \mathbf{Vect}_{\mathbb{F}}$$

from the *based link cobordism category* $\mathbf{Link}_p$. Objects of $\mathbf{Link}_p$ are oriented links in $\mathbb{R}^3$ containing p and morphisms are proper isotopy classes of collared, smoothly embedded link cobordisms in $\mathbb{R}^3 \times [0,1]$ containing the arc $p \times [0,1]$. In particular, two surfaces represent the same morphism if they differ by smooth isotopy fixing a neighborhood of the boundary and this arc pointwise. In order to define the functor Khr above, one first defines a functor from the *based diagrammatic link cobordism category* $\mathbf{Diag}_{\pi(p)}$. Objects of this category are equivalence classes of *based* movies in which each D_t contains $\pi(p)$. Any such movie can be expressed as a composition of elementary movies corresponding to Reidemeister moves, handle attachments, and planar isotopies, all supported away from $\pi(p)$. Two based movies are considered equivalent if they are related by obvious based versions of moves from before. To define a functor

$$Khr : \mathbf{Diag}_{\pi(p)} \to \mathbf{Vect}_{\mathbb{F}}$$

one then associates maps to elementary based movies and proceeds as before, noting that Jacobsson's work implies that equivalent based movies are assigned the same map. One then promotes this to a functor from $\mathbf{Link}_p$ by a straightforward adaptation of the ideas above.

Remark 2.16. It is clear that a similar procedure works for promoting any functor from **Diag** to $\mathbf{Spect}_{\mathbb{F}}$ to a functor from **Link** to $\mathbf{Spect}_{\mathbb{F}}$, and similarly for the based categories. With this in mind, we will be content to work solely in the diagrammatic categories in the rest of this paper.

3. Khovanov-Floer theories

In this section, we give a precise definition of a *Khovanov-Floer theory* (and its reduced variant) and describe what it means for such a theory to be *functorial.* The main challenge lies in setting up the right algebraic framework, as is illustrated by thinking about Kronheimer and Mrowka's spectral sequence in singular instanton knot homology. The difficulty is that their construction does not associate a filtered chain complex to a link diagram alone, but to a link diagram together with some auxiliary data (e.g. families of metrics and perturbations), so it is not immediately obvious in what sense the resulting spectral sequence gives an assignment of objects in $\mathbf{Spect}_{\mathbb{F}}$ to link diagrams. The same is true in Ozsváth and Szabó's work. Indeed, Kronheimer and Mrowka's construction assigns to a diagram D and a choice of data $\mathfrak{d}$ a filtered chain complex

$$C^{\mathfrak{d}}(D) = (C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))^4$$

and an isomorphism of vector spaces

$$q^{\mathfrak{d}} : Kh(D) \to E_2(C^{\mathfrak{d}}(D)).$$

Any two choices of auxiliary data $\mathfrak{d}, \mathfrak{d}'$ result in what one might call *quasi-isomorphic* constructions, in that there exists a filtered chain map

$$f : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D)$$

such that

$$E_2(f) = q^{\mathfrak{d}'} \circ (q^{\mathfrak{d}})^{-1}$$

(which implies that f is a quasi-isomorphism by the results of Subsection 2.1). So, really, one would like to say that what Kronheimer and Mrowka's construction assigns to a link diagram D is a *quasi-isomorphism class* of pairs $(C^{\mathfrak{d}}(D), q^{\mathfrak{d}})$. The algebraic framework introduced below is meant to make this idea meaningful.

Given a graded vector space V, we define a *V-complex* to be a pair (C, q), where C is a filtered chain complex and

$$q : V \to E_2(C)$$

is a grading-preserving isomorphism of vector spaces. Suppose (C, q) and (C', q') are V- and W-complexes, and let

$$T : V \to W$$

be a homogeneous degree k map of graded vector spaces. A *morphism from* (C, q) *to* (C', q') *which agrees on* E_2 *with* T is a degree k filtered chain map

$$f : C \to C'$$

such that

$$E_2(f) = q' \circ T \circ q^{-1}.$$

[4]We will often leave out the differential in the the notation for a chain complex.

Note that if f and g are two such morphisms, then $E_i(f) = E_i(g)$ for $i = 2$ and, therefore, for all $i \geq 2$ by Lemma 2.2. A *quasi-isomorphism* is a morphism from one V-complex to another which agrees on E_2 with the identity map on V.

Remark 3.1. Note that the existence of a quasi-isomorphism from (C, q) to (C', q') implies the existence of a quasi-isomorphism from (C', q') to (C, q) by Lemma 2.1.

For any two quasi-isomorphisms

$$f, g : (C, q) \to (C', q'),$$

we have that $E_i(f) = E_i(g)$ for all $i \geq 2$ by the discussion above. Moreover, given quasi-isomorphisms

$$f : (C, q) \to (C', q') \quad \text{and} \quad g : (C', q') \to (C'', q''),$$

we have that

$$E_i(g \circ f) = E_i(g) \circ E_i(f)$$

for all $i \geq 2$. In other words, the higher pages in the spectral sequences associated to quasi-isomorphic V-complexes are *canonically* isomorphic as vector spaces, and, since the $E_i(f)$ are chain maps, these higher pages are also canonically isomorphic *as chain complexes*. More precisely, for each $i \geq 2$, the collection of chain complexes (E_i, d_i) associated with representatives of a given quasi-isomorphism class $\mathcal{A}$ of V-complexes fits into a transitive system of chain complexes, from which one can extract an honest chain complex by taking the inverse limit. In summary, then, a quasi-isomorphism class $\mathcal{A}$ of V-complexes therefore determines a well-defined graded chain complex $(E_i(\mathcal{A}), d_i(\mathcal{A}))$ for each $i \geq 2$. This is the sense in which, for example, Kronheimer and Mrowka's construction provides an assignment of objects in $\mathbf{Spect}_{\mathbb{F}}$ to link diagrams.

Now suppose $\mathcal{A}$ is a quasi-isomorphism class of V-complexes and $\mathcal{B}$ is a quasi-isomorphism class of W-complexes, and let

$$T : V \to W$$

be a homogeneous degree k map of vector spaces. We will say that *there exists a morphism from $\mathcal{A}$ to $\mathcal{B}$ which agrees on E_2 with T* if there exists a morphism

$$f : (C, q) \to (C', q') \tag{5}$$

which agrees on E_2 with T for some representatives (C, q) of $\mathcal{A}$ and (C', q') of $\mathcal{B}$. The morphism in (5) gives rise to a homogeneous degree k map

$$E_i(\mathcal{A}) \to E_i(\mathcal{B}) \tag{6}$$

for each $i \geq 2$. Furthermore, this map is independent of the representative morphism in (5) in the sense that if (C'', q'') and (C''', q''') are other representatives of $\mathcal{A}$ and $\mathcal{B}$ and

$$f' : (C'', q'') \to (C''', q''')$$

is another morphism which agrees on E_2 with T, then the diagram

$$\begin{tikzcd}
E_i(C) \arrow[r, "E_i(f)"] \arrow[d] & E_i(C') \arrow[d] \\
E_i(C'') \arrow[r, "E_i(f')"'] & E_i(C''')
\end{tikzcd}$$

commutes for each $i \geq 2$, where the vertical arrows indicate the canonical isomorphisms between these higher pages. In summary, the existence of a morphism from $\mathcal{A}$ to $\mathcal{B}$ which agrees on E_2 with T *canonically* determines a chain map from $(E_i(\mathcal{A}), d_i(\mathcal{A}))$ to $(E_i(\mathcal{B}), d_i(\mathcal{B}))$ for all $i \geq 2$.

The discussion above shows that quasi-isomorphism classes of V-complexes behave exactly like honest filtered chain complexes with regard to the spectral sequences they induce. This will enable us to bypass the sort of technical difficulty mentioned at the beginning of this subsection for the spectral sequences defined by Kronheimer-Mrowka and Ozsváth-Szabó.

Finally, note that if (C, q) is a V-complex and (C', q') is a W-complex, then there is a natural tensor product in the form of a $(V \otimes W)$-complex $(C \otimes C', q \otimes q')$. This extends in the obvious way to a notion of tensor product between quasi-isomorphism classes of V- and W-complexes.

Below, we define the term *Khovanov-Floer theory*. In the definition, we are thinking of the vector space $Kh(D)$ as being singly-graded by some linear combination of the homological and quantum gradings. We will omit this linear combination from the notation. In practice, it will depend on the theory of interest: we will use the homological grading for the spectral sequence constructions of Kronheimer-Mrowka, Ozváth-Szabó, and Szabó, and the quantum grading for Lee's construction.

Definition 3.2. A *Khovanov-Floer theory* $\mathcal{A}$ is a rule which assigns to every link diagram D a quasi-isomorphism class of $Kh(D)$-complexes $\mathcal{A}(D)$, such that:

(1) if D' is obtained from D by a planar isotopy, then there exists a morphism

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the induced map from $Kh(D)$ to $Kh(D')$;

(2) if D' is obtained from D by a diagrammatic 1-handle attachments, then there exists a morphism

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the induced map from $Kh(D)$ to $Kh(D')$;

(3) for any two diagrams D, D', there exists a morphism

$$\mathcal{A}(D \sqcup D') \to \mathcal{A}(D) \otimes \mathcal{A}(D')$$

which agrees on E_2 with the standard isomorphism

$$Kh(D \sqcup D') \to Kh(D) \otimes Kh(D');$$

(4) for any diagram D of the unlink, $E_2(\mathcal{A}(D)) = E_\infty(\mathcal{A}(D))$.

A *reduced Khovanov-Floer theory* is defined almost exactly as above, except that all link diagrams are now based; planar isotopies and 1-handle attachments fix and avoid the basepoints, respectively; we replace all occurrences of Kh with Khr; and we replace condition (3) with the analogous condition for connected sum. Namely, for based diagrams D, D', and $D \# D'$ as at the end of Subsection 2.2, we require that there exists a morphism

$$\mathcal{A}(D \# D') \to \mathcal{A}(D) \otimes \mathcal{A}(D')$$

which agrees on E_2 with the standard isomorphism

$$Khr(D \# D') \to Khr(D) \otimes Khr(D')$$

described in that subsection.

Remark 3.3. Note that if $\mathcal{A}$ is a reduced Khovanov-Floer theory, then the assignment

$$D \mapsto \mathcal{A}(D \sqcup U),$$

with basepoint on U, naturally defines a Khovanov-Floer theory, via the relationship between reduced and unreduced Khovanov homology mentioned in Remark 2.10.

An immediate consequence of these definitions is that a Khovanov-Floer theory $\mathcal{A}$ assigns a *canonical* morphism of spectral sequences

$$\{(E_i(\mathcal{A}(D)), d_i(\mathcal{A}(D)) \to (E_i(\mathcal{A}(D'), E_i(\mathcal{A}(D'))\}_{i\geq 2}$$

to a movie corresponding to a planar isotopy or diagrammatic 1-handle attachment. Of course, we wish to show, in proving Theorem 1.5, that a Khovanov-Floer theory assigns a morphism of spectral sequences to *any* movie, such that equivalent movies are assigned the same morphism. This leads to the definition below.

Definition 3.4. A Khovanov-Floer theory $\mathcal{A}$ is *functorial* if, given a movie from D to D', there exists a morphism

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the induced map from $Kh(D)$ to $Kh(D')$.

Thus, a functorial Khovanov-Floer theory assigns a canonical morphism of spectral sequences

$$\{(E_i(\mathcal{A}(D)), d_i(\mathcal{A}(D)) \to (E_i(\mathcal{A}(D'), E_i(\mathcal{A}(D'))\}_{i\geq 2}$$

to any movie, which agrees on E_2 with the corresponding movie map on Khovanov homology. It follows that equivalent movies are assigned the same morphism since they are assigned the same map in Khovanov homology. In other words, the spectral sequence associated with a functorial Khovanov-Floer theory defines a functor from **Diag** to $\mathbf{Spect}_{\mathbb{F}}$ and, therefore, by Subsection 2.3, a functor

$$F : \mathbf{Diag} \to \mathbf{Spect}_{\mathbb{F}}$$

satisfying $SV_2 \circ F = Kh$. (In particular, the higher pages of the spectral sequence associated with a functorial Khovanov-Floer theory are link type invariants.) Thus, in order to prove Theorem 1.5, it suffices to prove the following theorem, which we do in the next section.

Theorem 3.5. *Every Khovanov-Floer theory is functorial.*

The rather simple conditions in the definition of a Khovanov-Floer theory may be thought of as a sort of *weak functoriality*. In practice, it is often relatively easy to verify that a theory satisfies these conditions (we will provide several such verifications in Section 5). In contrast, functoriality has not been verified for any of spectral sequence constructions that we know of. Reidemeister invariance has been established in a number of cases (including for the spectral sequences we consider in this paper), but the arguments are generally adapted to the particular theory under consideration. Our approach is more universal. In particular, Theorem 3.5 may be interpreted as saying that weak functoriality implies functoriality.

There is an obvious analogue of Definition 3.4 for reduced Khovanov-Floer theories, involving based movies and reduced Khovanov homology. The corresponding analogue of Theorem 3.5—that every reduced Khovanov-Floer theory is functorial—also holds by essentially the same proof.

4. Khovanov-Floer theories are functorial

This section is dedicated to proving Theorem 3.5 (and, therefore, Theorem 1.5).

Suppose $\mathcal{A}$ is a Khovanov-Floer theory. We will prove below that $\mathcal{A}$ is functorial. We first show that $\mathcal{A}$ assigns a canonical morphism of spectral sequences to the movie corresponding to *any* diagrammatic handle attachment (as opposed to only 1-handle attachments). This follows immediately from the proposition below.

Proposition 4.1. *If D' is obtained from D by a diagrammatic handle attachment, then there exists a morphism*

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the induced map from $\mathrm{Kh}(D)$ to $\mathrm{Kh}(D')$.

Proof of Proposition 4.1. The 1-handle case is part of the definition of a Khovanov-Floer theory. Suppose D' is obtained from D by a 0-handle attachment. Then $D' = D \sqcup U$. Thus, by condition (3) in Definition 3.2, there exists a morphism

$$\mathcal{A}(D) \otimes \mathcal{A}(U) \to \mathcal{A}(D') \tag{7}$$

which agrees on E_2 with the standard isomorphism

$$g_2 : Kh(D) \otimes Kh(U) \to Kh(D').$$

Condition (4) in Definition 3.2 says that

$$E_\infty(\mathcal{A}(U)) = E_2(\mathcal{A}(U)) \cong Kh(U).$$

It is then an easy consequence of Lemma 2.3 that the quasi-isomorphism class $\mathcal{A}(U)$ contains the *trivial* $Kh(U)$-complex $(Kh(U), id)$. It follows that there exists a morphism

$$\mathcal{A}(D) \to \mathcal{A}(D) \otimes \mathcal{A}(U) \tag{8}$$

which agrees on E_2 with the isomorphism

$$g_1 : Kh(D) \to Kh(D) \otimes Kh(U)$$

which sends x to $x \otimes 1$. Indeed, if (C, q) is a representative of $\mathcal{A}(D)$, then $(C \otimes Kh(U), q \otimes id)$ is a representative of $\mathcal{A}(D) \otimes \mathcal{A}(U)$ and the morphism

$$(C, q) \to (C \otimes Kh(D), q \otimes id)$$

which sends x to $x \otimes 1$ is a representative of the desired morphism in (8). Let f_1 and f_2 be representatives of the morphisms in (8) and (7), respectively. Then the composition $f_2 \circ f_1$ from a representative of $\mathcal{A}(D)$ to a representative of $\mathcal{A}(D')$ is a morphism which agrees on E_2 with the composition

$$g_2 \circ g_1 : Kh(D) \to Kh(D'),$$

and this latter composition is precisely the map on Khovanov homology associated to the 0-handle attachment. The 2-handle case is virtually identical. □

Next, we show that $\mathcal{A}$ assigns a canonical morphism of spectral sequences to the movie corresponding to a Reidemeister move. This follows immediately from the proposition below.

Proposition 4.2. *If D' is obtained from D by a Reidemeister move, then there exists a morphism*

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the standard isomorphism from $\mathrm{Kh}(D)$ *to* $\mathrm{Kh}(D')$.

Before proving Proposition 4.2, let us first assume this proposition is true and prove Theorem 3.5.

Proof of Theorem 3.5. Suppose M is a movie from D to D'. Express this movie as a composition

$$M = M_k \circ \cdots \circ M_1,$$

where each M_i is an elementary movie from a diagram D_i to a diagram D_{i+1}. Let f_i be a morphism from a representative of $\mathcal{A}(D_i)$ to a representative of $\mathcal{A}(D_{i+1})$ which agrees on E_2 with the corresponding map on Khovanov homology. For the elementary movies corresponding to planar isotopies, such maps exist by Definition 3.2. For those corresponding to diagrammatic handle attachments or Reidemeister moves, such maps exist by Propositions 4.1 and 4.2. The composition

$$f_k \circ \cdots \circ f_1$$

is therefore a morphism from a representative of $\mathcal{A}(D = D_1)$ to a representative of $\mathcal{A}(D' = D_{k+1})$ which agrees on E_2 with the map on Khovanov homology induced by this movie. This proves that $\mathcal{A}$ is functorial. □

It therefore only remains to prove Proposition 4.2. We break this verification into three lemmas—one for each type of Reidemeister move. The idea common to the proofs of all three lemmas is, as mentioned in the introduction, to arrange via movie moves that the Reidemeister move takes place amongst unknotted components. This idea was used by the third author in [25] in showing that a generic perturbation of Khovanov-Rozansky homology gives rise to a lower-bound on the slice genus.

Lemma 4.3. *Suppose D' is obtained from D by a Reidemeister I move. Then there exists a morphism*

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the standard isomorphism from $\mathrm{Kh}(D)$ *to* $\mathrm{Kh}(D')$.

Proof. Consider the link diagrams shown in Figure 3. The arrows in this figure are meant to indicate the fact that the movie represented by the sequence of diagrams

$$D = D_1, D_2, D_3, D_4 = D',$$

as indicated by the thin arrows, is equivalent to the movie consisting of the single Reidemeister I move from D to D', as indicated by the thick arrow. These two movies therefore induce the same map from $Kh(D)$ to $Kh(D')$. Thus, to prove Lemma 4.3, it suffices to prove that there exist morphisms

$$\mathcal{A}(D_1) \to \mathcal{A}(D_2) \tag{9}$$

$$\mathcal{A}(D_2) \to \mathcal{A}(D_3) \tag{10}$$

$$\mathcal{A}(D_3) \to \mathcal{A}(D_4) \tag{11}$$

which agree on E_2 with the corresponding maps on Khovanov homology. The top and bottom arrows in Figure 3 correspond to 0- and 1-handle attachments;

therefore, the morphisms in (9) and (11) exist by Proposition 4.1. It remains to show that the morphism in (10) exists.

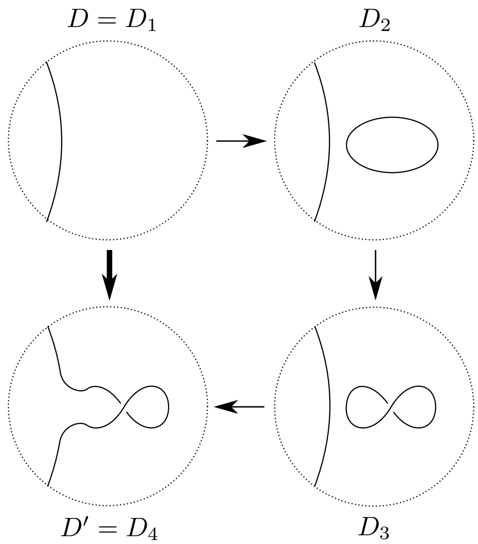

FIGURE 3. The diagrams $D = D_1, \ldots, D_4 = D'$. The movie indicated by the thin arrows is equivalent to the movie corresponding to the Reidemeister I move, indicated by the thick arrow.

Let U_0 and U_1 be the 0- and 1-crossing diagrams of the unknot in D_2 and D_3, so that $D_2 = D_1 \sqcup U_0$ and $D_3 = D_1 \sqcup U_1$. Thus, by condition (3) in Definition 3.2, there exist morphisms

$$\mathcal{A}(D_2) \to \mathcal{A}(D_1) \otimes \mathcal{A}(U_0) \tag{12}$$

$$\mathcal{A}(D_1) \otimes \mathcal{A}(U_1) \to \mathcal{A}(D_3) \tag{13}$$

which agree on E_2 with the standard isomorphisms

$$g_1 : Kh(D_2) \to Kh(D_1) \otimes Kh(U_0)$$

$$g_3 : Kh(D_1) \otimes Kh(U_1) \to Kh(D_3).$$

Condition (4) in Definition 3.2 says that

$$E_\infty(\mathcal{A}(U_i)) = E_2(\mathcal{A}(U_i)) \cong Kh(U_i)$$

for $i = 0, 1$, which implies, just as in the proof of Proposition 4.1, that the quasi-isomorphism class $\mathcal{A}(U_i)$ contains the trivial $Kh(U_i)$-complex $(Kh(U_i), id)$ for $i = 0, 1$. It follows immediately that there exists a morphism

$$\mathcal{A}(U_0) \to \mathcal{A}(U_1) \tag{14}$$

which agrees on E_2 with the standard isomorphism

$$g_2 : Kh(U_0) \to Kh(U_1)$$

associated to the Reidemeister I move relating these two diagrams of the unknot. Let f_1, f_2, and f_3 be representatives of the morphisms in (12), (14), and (13), respectively. Then the composition

$$f_3 \circ (id \otimes f_2) \circ f_1$$

from a representative of $\mathcal{A}(D_2)$ to a representative of $\mathcal{A}(D_3)$ is a morphism which agrees on E_2 with the composition

$$g_3 \circ (id \otimes g_2) \circ g_1 : Kh(D_2) \to Kh(D_3),$$

and this latter composition is equal to the isomorphism from $Kh(D_2)$ to $Kh(D_3)$ associated to the Reidemeister I move. It follows that the morphism in (10) exists. $\square$

Lemma 4.4. *Suppose D' is obtained from D by a Reidemeister II move. Then there exists a morphism*

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the standard isomorphism from $\mathrm{Kh}(D)$ *to* $\mathrm{Kh}(D')$.

Proof. Consider the link diagrams shown in Figure 4. The arrow from $D = D_1$ to D_2 represents two 0-handle attachments; the arrow from D_2 to D_3 represents a Reidemeister II move; and the arrow from D_3 to $D_4 = D'$ represents two 1-handle attachments. The movie represented by these thin arrows is equivalent to the movie from D to D' corresponding to the single Reidemeister II move indicated by the thick arrow. These two movies therefore induce the same map from $Kh(D)$ to $Kh(D')$. Thus, to prove Lemma 4.4, it suffices to prove that there exist morphisms

$$\mathcal{A}(D_1) \to \mathcal{A}(D_2) \tag{15}$$

$$\mathcal{A}(D_2) \to \mathcal{A}(D_3) \tag{16}$$

$$\mathcal{A}(D_3) \to \mathcal{A}(D_4) \tag{17}$$

which agree on E_2 with the corresponding maps on Khovanov homology. Since the top and bottom arrows in Figure 4 correspond to handle attachments, the morphisms in (15) and (17) exist by Proposition 4.1. It remains to show that the morphism in (16) exists.

Let U_0 and U_2 be the 0- and 2-crossing diagrams of the 2-component unlink in D_2 and D_3, so that $D_2 = D_1 \sqcup U_0$ and $D_3 = D_1 \sqcup U_2$. By condition (3) in Definition 3.2, there exist morphisms

$$\mathcal{A}(D_2) \to \mathcal{A}(D_1) \otimes \mathcal{A}(U_0) \tag{18}$$

$$\mathcal{A}(D_1) \otimes \mathcal{A}(U_2) \to \mathcal{A}(D_3) \tag{19}$$

which agree on E_2 with the standard isomorphisms

$$g_1 : Kh(D_2) \to Kh(D_1) \otimes Kh(U_0)$$

$$g_3 : Kh(D_1) \otimes Kh(U_1) \to Kh(D_3).$$

Condition (4) in Definition 3.2 says that

$$E_\infty(\mathcal{A}(U_i)) = E_2(\mathcal{A}(U_i)) \cong Kh(U_i)$$

for $i = 0, 2$, which implies as in the previous proof that the quasi-isomorphism class $\mathcal{A}(U_i)$ contains the trivial $Kh(U_i)$-complex $(Kh(U_i), id)$ for $i = 0, 2$. It follows immediately that there exists a morphism

$$\mathcal{A}(U_0) \to \mathcal{A}(U_2) \tag{20}$$

which agrees on E_2 with the standard isomorphism

$$g_2 : Kh(U_0) \to Kh(U_2)$$

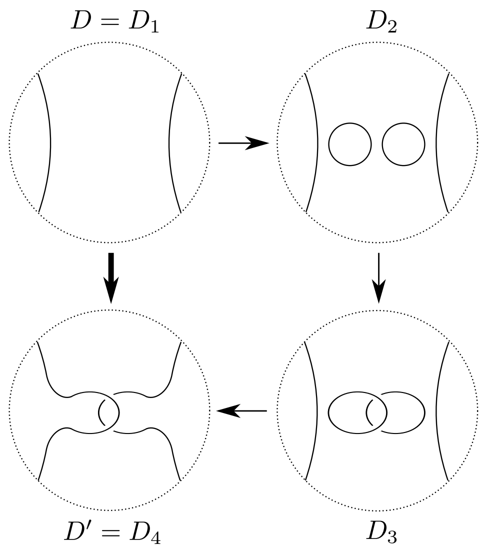

FIGURE 4. The diagrams $D = D_1, \ldots, D_4 = D'$. The movie indicated by the thin arrows is equivalent to the movie corresponding to the Reidemeister II move, indicated by the thick arrow.

associated to the Reidemeister II move relating these two diagrams of the unlink. Let f_1, f_2, and f_3 be representatives of the morphisms in (18), (20), and (19), respectively. Then the composition

$$f_3 \circ (id \otimes f_2) \circ f_1$$

from a representative of $\mathcal{A}(D_2)$ to a representative of $\mathcal{A}(D_3)$ is a morphism which agrees on E_2 with the composition

$$g_3 \circ (id \otimes g_2) \circ g_1 : Kh(D_2) \to Kh(D_3),$$

and this latter composition is equal to the isomorphism from $Kh(D_2)$ to $Kh(D_3)$ associated to the Reidemeister II move. It follows that the morphism in (16) exists. □

Lemma 4.5. *Suppose D' is obtained from D by a Reidemeister III move. Then there exists a morphism*

$$\mathcal{A}(D) \to \mathcal{A}(D')$$

which agrees on E_2 with the standard isomorphism from $\mathrm{Kh}(D)$ *to* $\mathrm{Kh}(D')$.

Proof. Consider the link diagrams shown in Figure 5. The arrow from $D = D_1$ to D_2 represents three 0-handle attachments; the arrow from D_2 to D_3 represents a sequence consisting of three Reidemeister II moves; the arrow from D_3 to D_4 represents a Reidemeister III move; the arrow from D_4 to D_5 represents three 1-handle attachments; and the arrow from D_5 to $D_6 = D'$ represents a sequence of three Reidemeister II moves. The movie represented by these thin arrows is equivalent to the movie from D to D' corresponding to the single Reidemeister III move indicated by the thick arrow. These two movies therefore induce the same map from $Kh(D)$ to $Kh(D')$. Thus, to prove Lemma 4.4, it suffices to prove that there exist morphisms

$$\mathcal{A}(D_1) \to \mathcal{A}(D_2) \tag{21}$$
$$\mathcal{A}(D_2) \to \mathcal{A}(D_3) \tag{22}$$
$$\mathcal{A}(D_3) \to \mathcal{A}(D_4) \tag{23}$$
$$\mathcal{A}(D_4) \to \mathcal{A}(D_5) \tag{24}$$
$$\mathcal{A}(D_5) \to \mathcal{A}(D_6) \tag{25}$$

which agree on E_2 with the corresponding maps on Khovanov homology. Since the top left and bottom right arrows in Figure 5 correspond to handle attachments, the morphisms in (21) and (24) exist by Proposition 4.1. The top right and bottom left arrows correspond to sequences of Reidemeister II moves, so the morphisms in (22) and (25) exist by Lemma 4.4. It remains to show that the morphism in (23) exists.

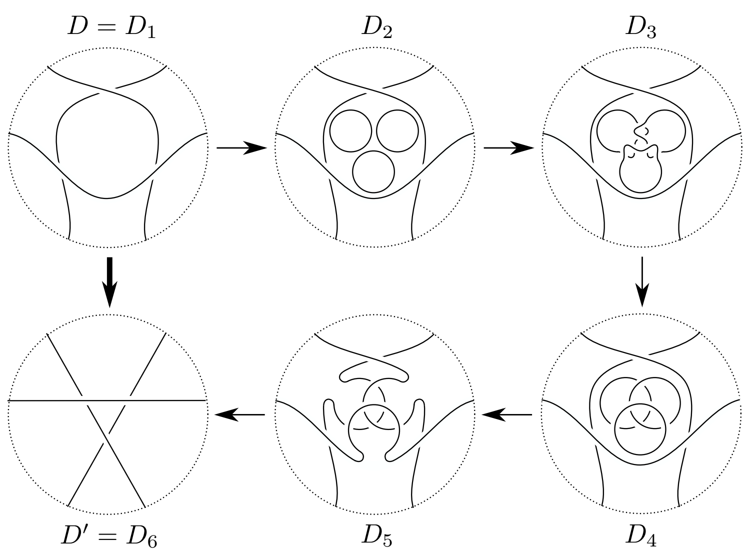

FIGURE 5. The diagrams $D = D_1, \ldots, D_6 = D'$. The movie indicated by the thin arrows is equivalent to the movie corresponding to the Reidemeister III move, indicated by the thick arrow.

Let U_a and U_b be the 6-crossing diagrams of the 3-component unlink in D_3 and D_4, so that $D_3 = D_1 \sqcup U_a$ and $D_4 = D_1 \sqcup U_b$. By condition (3) in Definition 3.2, there exist morphisms

$$\mathcal{A}(D_3) \to \mathcal{A}(D_1) \otimes \mathcal{A}(U_a) \tag{26}$$
$$\mathcal{A}(D_1) \otimes \mathcal{A}(U_b) \to \mathcal{A}(D_4) \tag{27}$$

which agree on E_2 with the standard isomorphisms

$$g_1 : Kh(D_3) \to Kh(D_1) \otimes Kh(U_a)$$
$$g_3 : Kh(D_1) \otimes Kh(U_b) \to Kh(D_4).$$

Condition (4) in Definition 3.2 says that

$$E_\infty(\mathcal{A}(U_i)) = E_2(\mathcal{A}(U_i)) \cong Kh(U_i)$$

for $i = a, b$, which implies as in the previous proof that the quasi-isomorphism class $\mathcal{A}(U_i)$ contains the trivial $Kh(U_i)$-complex $(Kh(U_i), id)$ for $i = a, b$. As before, it follows immediately that there exists a morphism

$$\mathcal{A}(U_a) \to \mathcal{A}(U_b) \tag{28}$$

which agrees on E_2 with the standard isomorphism

$$g_2 : Kh(U_a) \to Kh(U_b)$$

associated to the Reidemeister III move relating these two diagrams of the unlink. Let f_1, f_2, and f_3 be representatives of the morphisms in (26), (28), and (27), respectively. Then the composition

$$f_3 \circ (id \otimes f_2) \circ f_1$$

from a representative of $\mathcal{A}(D_3)$ to a representative of $\mathcal{A}(D_4)$ is a morphism which agrees on E_2 with the composition

$$g_3 \circ (id \otimes g_2) \circ g_1 : Kh(D_3) \to Kh(D_4),$$

and this latter composition is equal to the isomorphism from $Kh(D_3)$ to $Kh(D_4)$ associated to the Reidemeister III move. It follows that the morphism in (23) exists. $\square$

As mentioned in the previous section, the proof that reduced Khovanov-Floer theories are functorial proceeds in a virtually identical manner; we leave it to the reader to fill in the details.

5. Examples of Khovanov-Floer theories

In the first four subsections, we verify that the spectral sequence constructions of Kronheimer-Mrowka, Ozsváth-Szabó, Szabó, and Lee define Khovanov-Floer theories, proving Theorems 1.6, 1.7, and 1.8. All four verifications are formally very similar. Theorem 3.5 or its reduced analogue then imply that the associated spectral sequences define functors from **Link** or $\mathbf{Link}_p$ to $\mathbf{Spect}_{\mathbb{F}}$. As mentioned in the introduction, this also provides a new proof that Rasmussen's invariant is indeed a knot invariant.

In the fifth subsection, we describe some new deformations of the Khovanov complex which give rise to Khovanov-Floer theories, and potentially to new link type invariants (they of course give rise to functorial link type invariants, the question is whether these invariants are different from Khovanov homology).

5.1. **Kronheimer and Mrowka's spectral sequence.** Suppose D is a diagram for an oriented link $L \subset S^3 := \mathbb{R}^3 \cup \{\infty\}$, with crossings labeled $1, \ldots, n$. For each $I \in \{0,1\}^n$, let $L_I \subset S^3$ be a link whose projection to $\mathbb{R}^2$ is equal to D_I, and which agrees with L outside of n disjoint balls containing the "crossings" of L. For every pair $I <_1 I'$ of immediate successors, there is a standard 1-handle cobordism

$$S_{I,I'} \subset S^3 \times [0,1]$$

from L_I to $L_{I'}$ which is trivial outside the product of one of these balls with the interval. For any pair $I <_k J$ of tuples differing in k coordinates, choose a sequence $I = I_0 <_1 I_1 <_1 \cdots <_1 I_k = I'$ of immediate successors. Then the composition

$$S_{I,J} = S_{I_{k-1},I_k} \circ \cdots \circ S_{I_0,I_1}$$

defines a cobordism

$$S_{I,J} \subset S^3 \times [0,1]$$

from L_I to L_J which is independent of the sequence above, up to proper isotopy.

Given some auxiliary data $\mathfrak{d}$ (including a host of metric and perturbation data) Kronheimer and Mrowka construct [23] a chain complex $(C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))$, where

$$C^{\mathfrak{d}}(D) = \bigoplus_{I \in \{0,1\}^n} C^{\sharp}(\overline{L}_I)$$

and the differential $d^{\mathfrak{d}}(D)$ is a sum of maps

$$d_{I,J} : C^{\sharp}(\overline{L}_I) \to C^{\sharp}(\overline{L}_J)$$

over all pairs $I \leq J$ in $\{0,1\}^n$. Here, $C^{\sharp}(\overline{L}_I)$ refers to the unreduced singular instanton Floer chain group of $\overline{L}_I$ over $\mathbb{F}$. The map $d_{I,I}$ is the instanton Floer differential on $C^{\sharp}(\overline{L}_I)$, defined, *very roughly speaking*, by counting certain instantons on $S^3 \times \mathbb{R}$ with singularities along $\overline{L}_I \times \mathbb{R}$. More generally, $d_{I,J}$ is defined by counting points in parametrized moduli spaces of instantons on $S^3 \times \mathbb{R}$ with singularities along $S_{I,J}$, over a family of metrics and perturbations. We are abusing notation here, of course, as the vector spaces $C^{\sharp}(\overline{L}_I)$ and maps $d_{I,J}$ depend on $\mathfrak{d}$.

Kronheimer and Mrowka prove in [23] that the homology of this complex computes the unreduced singular instanton Floer homology of $\overline{L}$, as below.

Theorem 5.1. (Kronheimer-Mrowka [23]) *The homology $H_*(C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))$ is isomorphic to $I^{\sharp}(\overline{L})$.*

Note that the complex $(C^{\mathfrak{d}}(D), \partial^{\mathfrak{d}}(D))$ is a filtered complex with respect to the filtration coming from the *homological grading* defined by

$$\mathbf{h}(x) = I_1 + \cdots + I_n - n_-$$

for $x \in C^{\sharp}(\overline{L}_I)$. Since $d_{I,I}$ is the instanton Floer differential, the E_1 page of the associated spectral sequence is given by

$$E_1(C^{\mathfrak{d}}(D)) = \bigoplus_{I \in \{0,1\}^n} I^{\sharp}(\overline{L}_I).$$

Moreover, the spectral sequence differential $d_1(C^{\mathfrak{d}}(D))$ is the sum of the induced maps

$$(d_{I,I'})_* : I^{\sharp}(\overline{L}_I) \to I^{\sharp}(\overline{L}_{I'})$$

over all pairs $I <_1 I'$.

In [23, Section 8], Kronheimer and Mrowka establish isomorphisms

$$\Lambda^* V(D_I) \cong I^{\sharp}(\overline{L}_I)$$

which extend to an isomorphism of chain complexes

$$(CKh(D), d) \to (E_1(C^{\mathfrak{d}}(D)), d_1(C^{\mathfrak{d}}(D)))$$

that gives rise to an isomorphism

$$q^{\mathfrak{d}} : Kh(D) \to E_2(C^{\mathfrak{d}}(D)).$$

Moreover, they show that for any two sets of data $\mathfrak{d}$ and $\mathfrak{d}'$, there exists a filtered chain map

$$f : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D)$$

such that

$$E_2(f) = q^{\mathfrak{d}'} \circ (q^{\mathfrak{d}})^{-1}.$$

This is essentially the content of [23, Proposition 8.11] and the discussion immediately following it. In other words, Kronheimer and Mrowka's construction assigns to every link diagram D a quasi-isomorphism class of $Kh(D)$-complexes, with respect to the homological grading on $Kh(D)$. In fact, we claim the following.

Proposition 5.2. *Kronheimer-Mrowka's construction is a Khovanov-Floer theory.*

Proof. Let $\mathcal{A}(D)$ denote the quasi-isomorphism class of $Kh(D)$-complexes assigned to D in Kronheimer and Mrowka's construction. To prove the proposition, we simply check that $\mathcal{A}$ satisfies conditions (1)-(4) of Definition 3.2.

For condition (1), a planar isotopy ϕ taking D to D' determines a canonical filtered (in fact, grading-preserving) chain isomorphism

$$\psi_\phi : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D'),$$

where $\mathfrak{d}$ is the data pulled back from $\mathfrak{d}'$ via ϕ. Furthermore, it is clear that $E_1(\psi_\phi)$ agrees with the standard map

$$F_\phi : CKh(D) \to CKh(D')$$

associated to this isotopy in Khovanov homology, with respect to the natural identifications of the various chain complexes. It follows that ψ_ϕ represents a morphism from $\mathcal{A}(D)$ to $\mathcal{A}(D')$ which agrees on E_2 with the map induced on Khovanov homology, as desired.

For condition (2), suppose D' is obtained from D via a diagrammatic 1-handle attachment. Then there is a diagram $\tilde{D}$ with one more crossing than D and D', such that D is the 0-resolution of $\tilde{D}$ at this new crossing c and D' is the 1-resolution. For some choice of data $\tilde{\mathfrak{d}}$, we can realize the complex $C^{\tilde{\mathfrak{d}}}(\tilde{D})$ as the mapping cone of a degree 0 filtered chain map

$$T : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D'),$$

where $\mathfrak{d}$ and $\mathfrak{d}'$ are appropriate restrictions of $\tilde{\mathfrak{d}}$. This map T is given by the direct sum

$$T = \bigoplus_{I \leq J \in \{0,1\}^n} d_{I\times\{0\},J\times\{1\}},$$

of components of the differential $d^{\tilde{\mathfrak{d}}}(\tilde{D})$. (We are thinking of c as the $(n+1)^{\text{st}}$ crossing of $\tilde{D}$.) Then

$$E_1(T) : E_1(C^{\mathfrak{d}}(D)) \to E_1(C^{\mathfrak{d}'}(D'))$$

is given by the direct sum of the maps

$$(d_{I\times\{0\},I\times\{1\}})_* : I^\sharp(\overline{L}_I) \to I^\sharp(\overline{L}'_I)$$

over all $I \in \{0,1\}^n$. It follows from [23, Proposition 8.11] that these maps agree with the maps

$$\Lambda^* V(D_I) \to \Lambda^* V(D'_I)$$

associated to the 1-handle addition, via the natural identifications

$$\Lambda^* V(D_I) \cong I^\sharp(\overline{L}_I)$$
$$\Lambda^* V(D'_I) \cong I^\sharp(\overline{L}'_I)$$

described above. It follows that $E_1(T)$ agrees with the chain map

$$CKh(D) \to CKh(D')$$

associated to the 1-handle attachment, and, hence, that T represents a morphism from $\mathcal{A}(D)$ to $\mathcal{A}(D')$ which agrees on E_2 with the map induced on Khovanov homology, as desired.

For condition (3), it suffices to show that for some choices of data $\mathfrak{d}$, $\mathfrak{d}'$, $\mathfrak{d}''$, there is a degree 0 filtered chain map

$$C^{\mathfrak{d}''}(D \sqcup D') \to C^{\mathfrak{d}}(D) \otimes C^{\mathfrak{d}'}(D') \tag{29}$$

which agrees on E_2 with the standard isomorphism

$$Kh(D \sqcup D') \to Kh(D) \otimes Kh(D').$$

From the construction in [23, Subsection 5.5], there is an excision cobordism which gives rise to a quasi-isomorphism

$$C^\sharp(\overline{L}_I \sqcup \overline{L}'_J) \to C^\sharp(\overline{L}_I) \otimes C^\sharp(\overline{L}'_J) \tag{30}$$

for some such data and each pair $I, J \in \{0,1\}^n$. Moreover, since the induced isomorphism

$$I^\sharp(\overline{L}_I \sqcup \overline{L}'_J) \to I^\sharp(\overline{L}_I) \otimes I^\sharp(\overline{L}'_J)$$

is natural with respect to "split" cobordisms (see [23, Corollary 5.9]), it follows that this isomorphism agrees with the isomorphism

$$\Lambda^* V(D_I \sqcup D_J) \to \Lambda^* V(D_I) \otimes \Lambda^* V(D_J)$$

modulo the relevant natural identifications. The proof is then complete as long as one can show that the chain maps in (30) arise as the degree 0 components of a degree 0 filtered chain map as in (29). Although we do not give details, this can be arranged, defining the higher degree components of the chain map by counting instantons on the excision cobordism over higher dimensional families of metrics and perturbations, mimicking the arguments in [23, Section 6].

For condition (4), suppose D is a diagram of the unlink. Then its Khovanov homology is supported in homological degree 0. Hence, the spectral sequence collapses at the E_2 page. In particular, $E_2(\mathcal{A}(D)) = E_\infty(\mathcal{A}(D))$, as desired. $\square$

5.2. **Ozsváth and Szabó's spectral sequence.** Suppose D, L, and the L_I are exactly as in the previous subsection, except that L and the L_I are based at $p \in S^3$, and D is based at $\pi(p)$. Let a_j be an arc in a local neighborhood of the jth crossing of D as shown in Figure 6, and let b_j be a lift of a_j to an arc in S^3 with endpoints on L. The arc b_j lifts to a closed curve $\beta_j \subset -\Sigma(L)$, where $\Sigma(L)$ is the double branched cover of S^3 branched along L. There is a natural framing on the link

$$\mathbb{L} = \beta_1 \cup \cdots \cup \beta_n \subset -\Sigma(L)$$

such that $-\Sigma(L_I)$ is obtained by performing I_j-surgery on β_j for each $j = 1, \ldots, n$, for all $I \in \{0,1\}^n$.

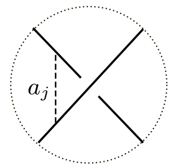

FIGURE 6. The arc a_j near the jth crossing, shown as a dashed segment.

Given some auxiliary data $\mathfrak{d}$ (including a pointed Heegaard multi-diagram *subordinate* to the framed link $\mathbb{L}$ and a host of complex-analytic data), Ozsváth and Szabó construct [28] a chain complex $(C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))$, where

$$C^{\mathfrak{d}}(D) = \bigoplus_{I\in\{0,1\}^n} \widehat{CF}(-\Sigma(L_I))$$

and the differential $d^{\mathfrak{d}}(D)$ is a sum of maps

$$d_{I,J} : \widehat{CF}(-\Sigma(L_I)) \to \widehat{CF}(-\Sigma(L_J))$$

over all pairs $I \leq J$ in $\{0,1\}^n$. Here, $\widehat{CF}(-\Sigma(L_I))$ refers to the Heegaard Floer chain group of $-\Sigma(L_I)$. The map $d_{I,I}$ is the usual Heegaard Floer differential on $\widehat{CF}(-\Sigma(L_I))$, defined by counting pseudo-holomorphic disks in the symmetric product of a Riemann surface. More generally, $d_{I,J}$ is defined by counting pseudo-holomorphic polygons. Again, we are abusing notation here, as the vector spaces $\widehat{CF}(-\Sigma(L_I))$ and maps $d_{I,J}$ depend on $\mathfrak{d}$.

Ozsváth and Szabó prove in [28] that the homology of this complex computes the Heegaard Floer homology of $-\Sigma(L)$; that is:

Theorem 5.3. *The homology $H_*(C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))$ is isomorphic to $\widehat{\mathrm{HF}}(-\Sigma(L))$.*

As in the previous subsection, this complex $(C^{\mathfrak{d}}(D), \partial^{\mathfrak{d}}(D))$ is filtered with respect to the obvious homological grading. Since $d_{I,I}$ is the Heegaard Floer differential, the E_1 page of the associated spectral sequence is given by

$$E_1(C^{\mathfrak{d}}(D)) = \bigoplus_{I\in\{0,1\}^n} \widehat{HF}(-\Sigma(L_I)).$$

Moreover, the spectral sequence differential $d_1(C^{\mathfrak{d}}(D))$ is the sum of the induced maps

$$(d_{I,I'})_* : \widehat{HF}(-\Sigma(L_I)) \to \widehat{HF}(-\Sigma(L_{I'}))$$

over all pairs $I <_1 I'$.

Below, we argue that Ozsváth and Szabó's construction assigns to D a quasi-isomorphism class of $Khr(D)$-complexes.

In general, the Heegaard Floer homology of a 3-manifold Y admits an action by $\Lambda^* H_1(Y)$. For each $I \in \{0,1\}^n$, the Floer homology $\widehat{HF}(-\Sigma(L_I))$ is a free module over

$$\Lambda^* H_1(-\Sigma(L_I))$$

of rank one, generated by the unique element in the top Maslov grading. In particular, there is a canonical identification

$$\widehat{HF}(-\Sigma(L_I)) \cong \Lambda^* H_1(-\Sigma(L_I)). \tag{31}$$

Suppose x is the component of D_I containing the basepoint $\pi(p)$. Given any other component y, let $\eta_{x,y}$ be an arc with endpoints on L_I which projects to an arc from x to y. The map

$$V(D_I)/(x) \to H_1(-\Sigma(L_I))$$

which sends a component y to the homology class of the lift of $\eta_{x,y}$ to the branched double cover clearly gives rise to an isomorphism

$$\Lambda^*(V(D_I)/(x)) \to \widehat{HF}(-\Sigma(L_I))$$

via the identification in (31). Moreover, Ozsváth and Szabó show that the direct sum of these isomorphisms gives rise to an isomorphism of chain complexes

$$(CKhr(D), d) \to (E_1(C^{\mathfrak{d}}(D)), d_1(C^{\mathfrak{d}}(D))).$$

This isomorphism then gives rise to an isomorphism

$$q^{\mathfrak{d}} : Khr(D) \to E_2(C^{\mathfrak{d}}(D)).$$

It follows from the work in [2, 32] and naturality properties of the $\Lambda^* H_1$-action that for any two sets of data $\mathfrak{d}$ and $\mathfrak{d}'$, there exists a filtered chain map

$$f : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D)$$

such that

$$E_2(f) = q^{\mathfrak{d}'} \circ (q^{\mathfrak{d}})^{-1}.$$

This shows that Ozsváth and Szabó's construction assigns to every based link diagram D a quasi-isomorphism class of $Khr(D)$-complexes, with respect to the homological grading on $Khr(D)$. In fact, we claim the following.

Proposition 5.4. *Ozsváth and Szabó's construction is a reduced Khovanov-Floer theory.*

Proof. Let $\mathcal{A}(D)$ denote the quasi-isomorphism class of $Khr(D)$-complexes assigned to D in Ozsváth and Szabó's construction. We verify below that $\mathcal{A}$ satisfies the reduced analogues of conditions (1)-(4) of Definition 3.2.

For condition (1), a planar isotopy ϕ taking D to D' determines a canonical filtered (in fact, grading-preserving) chain isomorphism

$$\psi_\phi : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D'),$$

where $\mathfrak{d}$ is the data pulled back from $\mathfrak{d}'$ via ϕ, just as in the instanton case. Furthermore, it is clear that $E_1(\psi_\phi)$ agrees with the standard map

$$F_\phi : CKhr(D) \to CKhr(D')$$

associated to this isotopy in reduced Khovanov homology, with respect to the natural identifications of the various chain complexes. It follows that ψ_ϕ represents a morphism from $\mathcal{A}(D)$ to $\mathcal{A}(D')$ which agrees on E_2 with the map induced on reduced Khovanov homology, as desired.

For condition (2), Suppose D' is obtained from D via a 1-handle attachment. Let $\tilde{D}$ be a diagram with one more crossing than D and D', such that D is the 0-resolution of $\tilde{D}$ at this crossing and D' is the 1-resolution, as in the proof of Proposition 5.2. Following that proof, we can realize the complex $C^{\tilde{\mathfrak{d}}}(\tilde{D})$ as the mapping cone of a degree 0 filtered chain map

$$T : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D'),$$

for some choice of data $\tilde{\mathfrak{d}}$ and the appropriate restrictions $\mathfrak{d}$ and $\mathfrak{d}'$. As before, T is given by the direct sum

$$T = \bigoplus_{I \leq J \in \{0,1\}^n} d_{I\times\{0\},J\times\{1\}},$$

of components of the differential $d^{\tilde{\mathfrak{d}}}(\tilde{D})$, and

$$E_1(T) : E_1(C^{\mathfrak{d}}(D)) \to E_1(C^{\mathfrak{d}'}(D'))$$

is the direct sum of the maps

$$(d_{I\times\{0\},I\times\{1\}})_* : \widehat{HF}(-\Sigma(L_I)) \to \widehat{HF}(-\Sigma(L'_I))$$

over all $I \in \{0,1\}^n$. It is easy to see that these maps agree with the maps

$$\Lambda^*(V(D_I)/(x)) \to \Lambda^*(V(D'_I)/(x'))$$

associated to the 1-handle attachment, via the natural identifications

$$\Lambda^*(V(D_I)/(x)) \cong \widehat{HF}(-\Sigma(L_I))$$
$$\Lambda^*(V(D'_I)/(x')) \cong \widehat{HF}(-\Sigma(L'_I)),$$

where x and x' are the components of D_I and D'_I containing the basepoint $\pi(p)$. It follows that $E_1(T)$ agrees with the chain map

$$CKhr(D) \to CKhr(D')$$

associated to the 1-handle attachment, and, hence, that T represents a morphism from $\mathcal{A}(D)$ to $\mathcal{A}(D')$ which agrees on E_2 with the map induced on reduced Khovanov homology, as desired.

For condition (3), it suffices as in the instanton Floer case to show that for some sets of data $\mathfrak{d}$, $\mathfrak{d}'$, $\mathfrak{d}''$, there is a degree 0 filtered chain map

$$C^{\mathfrak{d}''}(D\#D') \to C^{\mathfrak{d}}(D) \otimes C^{\mathfrak{d}'}(D')$$

which agrees on E_2 with the standard isomorphism

$$Khr(D\#D') \to Khr(D) \otimes Khr(D'),$$

where D, D' and $D\#D'$ are based diagrams as at the end of Subsection 2.2. But, given the Heegaard multi-diagrams encoded by $\mathfrak{d}$ and $\mathfrak{d}'$, one can simply take an appropriate connected sum to produce a multi-diagram giving rise to a complex $C^{\mathfrak{d}''}(D\#D')$ which is isomorphic to the tensor product

$$C^{\mathfrak{d}}(D) \otimes C^{\mathfrak{d}'}(D')$$

by an isomorphism which agrees on E_2 with the map on reduced Khovanov homology (see [2, Lemma 3.4]).

For condition (4), suppose D is a diagram of the unlink. Then its reduced Khovanov homology is supported in homological degree 0. Hence, the spectral sequence collapses at the E_2 page. In particular, $E_2(\mathcal{A}(D)) = E_\infty(\mathcal{A}(D))$, as desired. $\square$

5.3. **Szabó's geometric spectral sequence.** Suppose D is a link diagram as in Subsection 5.1. Given auxiliary data $\mathfrak{d}$ consisting of a *decoration* of D, Szabó defines in [34] a chain complex $(C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))$, where

$$C^{\mathfrak{d}}(D) = \bigoplus_{I \in \{0,1\}^n} CKh(D_I)$$

and the differential $d^{\mathfrak{d}}(D)$ is a sum of maps

$$d_{I,J} : \Lambda^* V(D_I) \to \Lambda^* V(D_J)$$

over all pairs $I \leq J$ in $\{0,1\}^n$. The maps $d_{I,I}$ are identically zero. For $I <_1 I'$, the map $d_{I,I'}$ is the usual merge or split map used to define the Khovanov differential; in particular, it does not depend on the decoration $\mathfrak{d}$. The maps $d_{I,J}$ are also defined combinatorially, but do depend on $\mathfrak{d}$. It is an interesting question what the homology of this complex computes. Szabó conjectures the following in [34].

Conjecture 5.5. *The homology $H_*(C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))$ is isomorphic to $\oplus^2 \widehat{\mathrm{HF}}(-\Sigma(L))$.*

The complex $(C^{\mathfrak{d}}(D), d^{\mathfrak{d}}(D))$ is obviously filtered with respect to the homological grading. By construction,

$$(CKh(D), d) = (E_1(C^{\mathfrak{d}}(D)), d_1(C^{\mathfrak{d}}(D))),$$

so that

$$Kh(D) = E_2(C^{\mathfrak{d}}(D))$$

on the nose. We may therefore define each $q^{\mathfrak{d}}$ to be the identity map. Szabó shows that for any two sets of data $\mathfrak{d}$ and $\mathfrak{d}'$, there exists a filtered chain map

$$f : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D)$$

which is equal to the identity map on E_2. In particular, Szabó's construction assigns to every link diagram D a quasi-isomorphism class of $Kh(D)$-complexes, with respect to the homological grading on $Kh(D)$. As before, we claim the following.

Proposition 5.6. *Szabó's construction is a Khovanov-Floer theory.*

Proof. Let $\mathcal{A}(D)$ denote the quasi-isomorphism class of $Kh(D)$-complexes assigned to D in Szabó's construction. The proof of this proposition is again just a verification that $\mathcal{A}$ satisfies conditions (1)-(4) of Definition 3.2.

For condition (1), we proceed exactly as in the previous two subsections.

For condition (2), we also proceed as in those subsections. Let D, D', and $\tilde{D}$ be diagrams described previously. We may choose a decoration $\tilde{\mathfrak{d}}$ for $\tilde{D}$ which restricts to decorations $\mathfrak{d}$ and $\mathfrak{d}'$ for D and D'. It follows from the *Extension Formula* in [34, Definition 2.5] that we can realize the complex $C^{\tilde{\mathfrak{d}}}(\tilde{D})$ as the mapping cone of the degree 0 filtered chain map

$$T : C^{\mathfrak{d}}(D) \to C^{\mathfrak{d}'}(D')$$

given by the direct sum

$$T = \bigoplus_{I \leq J \in \{0,1\}^n} d_{I\times\{0\}, J\times\{1\}},$$

of components of the differential $d^{\tilde{\mathfrak{d}}}(\tilde{D})$. Then

$$E_1(T) : E_1(C^{\mathfrak{d}}(D)) \to E_1(C^{\mathfrak{d}'}(D'))$$

is given by the direct sum of the maps

$$d_{I\times\{0\},I\times\{1\}} : \Lambda^*V(D_I) \to \Lambda^*V(D'_I)$$

over all $I \in \{0,1\}^n$. But these are precisely the maps associated to the 1-handle attachment. Thus, $E_1(T)$ agrees with the chain map

$$CKh(D) \to CKh(D')$$

associated to the 1-handle attachment. It follows that T represents a morphism from $\mathcal{A}(D)$ to $\mathcal{A}(D')$ which agrees on E_2 with the map induced on Khovanov homology, as desired.

For condition (3), suppose $\mathfrak{d}$ and $\mathfrak{d}'$ are decorations for D and D', and let $\mathfrak{d}''$ be the corresponding decoration for $D \sqcup D'$. Then the *Disconnected Rule* [34, Definition 2.7] implies that

$$C^{\mathfrak{d}''}(D \sqcup D') = C^{\mathfrak{d}}(D) \otimes C^{\mathfrak{d}'}(D')$$

as complexes. It follows immediately that $\mathcal{A}$ satisfies condition (3).

For condition (4), suppose D is a diagram of the unlink. Then its Khovanov homology is supported in homological degree 0. Hence, the spectral sequence collapses at the E_2 page. In particular, $E_2(\mathcal{A}(D)) = E_\infty(\mathcal{A}(D))$, as desired. □

5.4. **Lee's spectral sequence.** Let D be a link diagram as in the previous subsections. In [24], Lee defined a perturbation of the Khovanov complex of D which, over $\mathbb{Q}$, gives rise to a spectral sequence with E_2 page the Khovanov homology $Kh(D)$ and converging to $(\mathbb{Q} \oplus \mathbb{Q})^k$, where k is the number of components of D. When D is a knot diagram, Rasmussen's numerical invariant $s_{\mathbb{Q}}$ [29] mentioned in the introduction may be defined as the average of the quantum gradings on the two summands of the E_∞ page of this spectral sequence. This invariant defines a homomorphism from the smooth concordance group to $\mathbb{Z}$, and provides a lower bound on the smooth slice genus.

In [4], Bar-Natan defined a version of Lee's construction for coefficients in $\mathbb{F}$, with the corresponding properties as above. Roughly speaking, Bar-Natan's theory is built from the $(1+1)$-dimensional TQFT associated with the Frobenius algebra $\mathbb{F}[x]/(x^2+x)$ while Lee's theory corresponds to the Frobenius algebra $\mathbb{Q}[x]/(x^2-1)$.

Bar-Natan's construction assigns to D a chain complex $(C(D), d^{BN})$, where

$$C(D) = \bigoplus_{I\in\{0,1\}^n} CKh(D_I)$$

and the differential d^{BN} is a sum of maps

$$d^{BN}_{I,I'} : \Lambda^*V(D_I) \to \Lambda^*V(D_{I'})$$

over all pairs $I <_1 I' \in \{0,1\}^n$. Here,

$$d^{BN}_{I,I'} = d_{I,I'} + d'_{I,I'},$$

where $d_{I,I'}$ is the standard merge or split map used to define the Khovanov differential, and $d'_{I,I'}$ is a map which raises the quantum grading by 2. Turner proves the following in [35].

Theorem 5.7. *The homology $H_*(C(D), d^{BN}) \cong (\mathbb{F} \oplus \mathbb{F})^k$, where k is the number of components of D.*

Note that $(C(D), d^{BN})$ is a filtered complex with respect to the quantum grading. Furthermore,

$$Kh(D) = E_1(C(D)) = E_2(C(D))$$

for the associated spectral sequence. It is thus clear that Bar-Natan's construction assigns to a link diagram D a quasi-isomorphism class of $Kh(D)$-complexes, with respect to the quantum grading on $Kh(D)$. Moreover, we have the following.

Proposition 5.8. *Bar-Natan's construction is a Khovanov-Floer theory.*

Proof. The proof proceeds almost exactly as in the previous subsections, but is even easier; we omit it here. □

As mentioned above, if D is a knot diagram, then Rasmussen's invariant $s_{\mathbb{F}}$ can be defined as the average of quantum gradings of the two summands of

$$E_\infty(C(D)) \cong \mathbb{F} \oplus \mathbb{F}.$$

A priori, this average depends on the diagram D. The fact that Bar-Natan's spectral sequence is functorial provides an independent proof that this average is, in fact, a knot invariant.

5.5. **New knot invariants.** In addition to gathering known spectral sequences from Khovanov homology under the umbrella of our Khovanov-Floer formalism, there is an opportunity to search for combinatorial perturbations of the Khovanov differential which give rise to Khovanov-Floer theories. Our main result shows that any such perturbation gives a spectral sequence which is a functorial knot invariant. Szabó's geometric spectral sequence [34] and Lee's deformation [24] provide examples in which the resulting spectral sequence may be non-trivial.

In this subsection we give three examples of combinatorial perturbations in order to stimulate further work in classifying such perturbations and in computing their spectral sequences. We do not know if, for example, the spectral sequence of any perturbation that we give here necessarily collapses at the E_2 page for all links.

Suppose that $I, J \in \{0,1\}^n$ such that $I <_k J$, and choose a sequence of immediate successors

$$I = I_0 <_1 I_1 <_1 I_2 <_1 \cdots <_1 I_k = J.$$

For a planar diagram D with crossings $1, \ldots, n$, this sequence defines a map

$$d_{I,J} = d_{I_{k-1},I_k} \circ \cdots \circ d_{I_0,I_1} : \Lambda^* V(D_I) \to \Lambda^* V(D_J).$$

Note that this map does not depend on the choice of sequence since 2-dimensional faces in the Khovanov cube commute.

Now we define the endomorphism

$$d_k = \bigoplus_{I <_k J} d_{I,J} : CKh(D) \to CKh(D)$$

for each $k \geq 1$. Note that each d_k preserves the quantum grading and shifts the homological grading by k, and that d_1 is the Khovanov differential. Finally, for any sequence $\underline{\mathbf{a}} = (a_1, a_2, a_3, a_4, \ldots)$ where $a_i \in \mathbb{F}$ for all $i \geq 1$ and $a_1 = 1$ we define the endomorphism

$$d_{\underline{\mathbf{a}}} = \bigoplus_{k \geq 1} a_k d_k : CKh(D) \to CKh(D).$$

We shall check that $d_{\underline{\mathbf{a}}}^2 = 0$ and leave it as an (easy) exercise for the reader to verify that this defines a Khovanov-Floer theory with a homological filtration and a quantum grading.

The key ingredient in the check is that $d_{I,K} = d_{J,K} \circ d_{I,J}$ for any $I \leq J \leq K$. For convenience if k is an odd integer then set $a_{k/2} = 0$ and set the binomial coefficient

$$\begin{pmatrix} k \\ k/2 \end{pmatrix} = 0.$$

We have

$$\begin{aligned}
d_{\underline{\mathbf{a}}}^2 &= \bigoplus_{i,j \geq 1} a_j d_j \circ a_i d_i = \bigoplus_{i,j \geq 1} (a_j a_i) d_j \circ d_i \\
&= \bigoplus_{\substack{I <_i J <_j K \\ i,j \geq 1}} (a_j a_i) d_{J,K} \circ d_{I,J} = \bigoplus_{\substack{I <_i J <_j K \\ i,j \geq 1}} (a_j a_i) d_{I,K} \\
&= \bigoplus_{\substack{I <_k K \\ k \geq 2, k-1 \geq j \geq 1}} (a_j a_{k-j}) \begin{pmatrix} k \\ j \end{pmatrix} d_{I,K} \\
&= \bigoplus_{\substack{I <_k K \\ k \geq 2, k/2 > j \geq 1}} \left(2 \begin{pmatrix} k \\ j \end{pmatrix} (a_j a_{k-j}) + \begin{pmatrix} k \\ k/2 \end{pmatrix} (a_{k/2} a_{k/2}) \right) d_{I,K} = 0.
\end{aligned}$$

This concludes the first example. For the second example, consider the same set-up of a planar diagram D with crossings $1, \ldots, n$. Now look for all pairs $(I, J) \in \{0,1\}^n$ such that $I <_2 J$ and such that if $I <_1 K <_1 J$ then the movie represented by the sequence

$$D_I, D_K, D_J$$

consisting of two 1-handle attachments describes a cobordism which is the union of a twice-punctured torus with some annuli. We call such a pair (I, J) a *ladybug configuration* and write the set of all ladybug configurations as $\mathbf{L}$.

For $(I, J) \in \mathbf{L}$ we wish to define a map

$$d'_{I,J} : \Lambda^* V(D_I) \to \Lambda^* V(D_J).$$

To do this, we first identify D_I with D_J (and so $\Lambda^* V(D_I)$ with $\Lambda^* V(D_J)$) by identifying circles which are part of the same connected component of the cobordism. Then the map $d'_{I,J}$ is defined to be wedging with the generator of $\Lambda^* V(D_I)$ corresponding to a boundary component of the genus 1 part of the cobordism.

Now define the endomorphism

$$d_L = d \oplus \bigoplus_{(I,J) \in \mathbf{L}} d'_{I,J} : CKh(D) \to CKh(D),$$

where d is the Khovanov differential. Note that, as in the case of $d_{\underline{\mathbf{a}}}$, we have that d_L preserves the quantum grading.

Again, once it is verified that $d_L^2 = 0$, it is an easy exercise to see that this defines a Khovanov-Floer theory. The check that $d_L^2 = 0$ is combinatorial. Explicitly, for any $I <_3 J$ we need to verify that

$$\bigoplus_{I <_1 K <_2 J} d'_{K,J} \circ d_{I,K} \oplus \bigoplus_{I <_2 K <_1 J} d_{K,J} \circ d'_{I,K} : \Lambda^* V(D_I) \to \Lambda^* V(D_J) = 0,$$

and for any $I <_4 J$ we need to verify that

$$\bigoplus_{I<_2K<_2J} d'_{K,J} \circ d'_{I,K} : \Lambda^* V(D_I) \to \Lambda^* V(D_J) = 0.$$

Both checks may be made along the same lines as the checks in Szabo's [34], although this case is easier since there is no auxiliary data of a *decoration*. The first check should be carried out for each 3-dimensional configuration, and the second for each 4-dimensional configuration. We leave these checks for the reader to verify.

Finally we very briefly give an example that makes use of a quantum rather than a homological filtration. The idea can be summarized simply as replacing the "saddle" differential by the sum of a saddle and a dotted saddle (in the sense of Bar-Natan [4]). Then all differentials raise the homological grading by 1, while respecting a quantum filtration.

Explicitly, if $I <_1 J$ we define components of the deformed differential as

$$d'_{I,J} = d_{I,J} + d_{I,J} \wedge x$$

where $d_{I,J}$ is the Khovanov differential, and where by $\wedge x$ we mean post composition by wedging with a generator x corresponding to one of the (possibly two) circles of the resolution J in the boundary of the pair of pants cobordism component. This is independent of the choice of x (as the "dotting" formalism above suggests).

Remark 5.9. The first deformation above in the case $\underline{\mathbf{a}} = (1, 1, 1, \ldots)$ was studied independently by Juhász and Marengon. In [19, Section 6], they also show that the isomorphism class of the resulting spectral sequence is a link type invariant.

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