Prof. Norbert Peyerimhoff, Durham University 12/2/2014

Algebraic Geometry III/IV

Problems, set 4. To be handed in on **Wednesday, 19 February 2014**, in the lecture.

Exercise 6. The aim of this exercise is to prove that the Riemann sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

is a one-dimensional complex manifold. We first introduce the coordinate charts via *stereographic projections* from the north pole and the south pole of S^2: Let $n = (0, 0, 1), s = (0, 0, -1) \in S^2$. For any two distinct points $p, q \in \mathbb{R}^3$ let $L_{pq} \subset \mathbb{R}^3$ denote the straight Euclidean line through p and q. Let $U_1 = S^2 \backslash \{n\}$ and $U_2 = S^2 \backslash \{s\}$. Let $V_1 = \mathbb{R}^2 \times \{0\} \cong \mathbb{C}$ via the identification $(x, y, 0) \mapsto z = x + yi$ and $\phi_1 : U_1 \to V_1$ be defined by $\phi_1(p) = L_{np} \cap V_1$. Let $V_2 = \mathbb{R}^2 \times \{0\} \cong \mathbb{C}$ via the identification $(x, y, 0) \mapsto z = x - yi$ and $\phi_2 : U_2 \to V_2$ be defined by $\phi_2(p) = L_{sp} \cap V_2$.

(a) For $(x, y, z) \in S^2$, calculate explicitly $\phi_1(x, y, z) \in \mathbb{C}$ and $\phi_2(x, y, z) \in \mathbb{C}$.

(b) For $z \in \mathbb{C}$, calculate explicitly the inverse maps $\phi_1^{-1}(z) \in S^2$ and $\phi_2^{-1}(z) \in S^2$.

(c) Calculate the coordinate changes

$$\phi_2 \circ \phi_1^{-1}, \quad \phi_1 \circ \phi_2^{-2} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\},$$

and check that they are holomorphic maps.

This shows that S^2 is a Riemann surface, since $S^2 = U_1 \cup U_2$.

Exercise 7. Let M_1, M_2 be two Riemann surfaces. A bijiective map $f : M_1 \to M_2$ is called a *biholomorphic map* if both $f : M_1 \to M_2$ and $f^{-1} : M_2 \to M_1$ are both holomorphic maps.

(a) Find a biholomorphic map $f : \mathbb{P}^1_{\mathbb{C}} \to S^2$, and show that f is biholomorphic.

(b) You learnt in the lectures that every non-singular projective conic is projectively equivalent to $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ with $F(X,Y,Z) = X^2 - YZ$. Show that the map

$$g : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}, \quad g([a,b]) = [ab, a^2, b^2]$$

is a biholomorphic map between $\mathbb{P}^1_{\mathbb{C}}$ and C_F.