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Algebraic Geometry III/IV

Problems, set 5.

Exercise 8. In this exercise, we introduce a parametrisation for non-singular projective cubics which explains their global topological structures as real 2-dimensional tori (genus $g = 0$), and which also makes their group structures completely obvious. Students who took "Elliptic Functions III" may find this exercise particularly appealing. This exercise needs a considerable amount of preparation, but it is really worth to work it through, because this exercise gives wonderful insights...

Firstly, we state that every non-singular cubic $C \subset \mathbb{P}^2_{\mathbb{C}}$ can be transformed via a projective transformation into a curve C_F with

$$F(X,Y,Z) = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3,$$

where $g_2, g_3 \in \mathbb{C}$ satisfy $g_2^3 - 27g_3^2 \neq 0$ (otherwise C_F is not non-singular). This form is called the *Weierstraß normal form* of C and you will prove this normal form in next week's exercise sheet.

Secondly, there is a parametrisation of C_F via the *Weierstraß $\wp$-function* associated to a lattice $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ (with $\lambda_1, \lambda_2 \in \mathbb{C}$ linear independent over $\mathbb{R}$). This function is defined as follows:

$$\wp(z) = \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

We will give the parametrisation of C_F via $\wp(z)$ very shortly.

Thirdly, we need to inroduce the notion of elliptic functions and mention some basic properties of them: An *elliptic function* f is a meromorphic function on $\mathbb{C}$ which is doubly periodic with respect to a lattice Λ, i.e.,

$$f(z+\omega) = f(z) \quad \forall\, \omega \in \Lambda.$$

All elliptic functions with respect to a fixed lattice Λ form a $\mathbb{C}$-vector space. Examples of elliptic functions are $\wp_\Lambda(z)$ and its derivative $\wp'_\Lambda(z)$. They satisfy the following differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda),$$

where

$$g_2(\Lambda) = 60 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^4}, \tag{1}$$

$$g_3(\Lambda) = 140 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^6}. \tag{2}$$

This differential equation is the key to the parametrisation of C_F.

Next we discuss some important basic properties: Let f be an elliptic function with respect to Λ and $\mathcal{F} = \{s\lambda_1 + t\lambda_2 \mid 0 \le s,t < 1\} \subset \mathbb{C}$. (The parallellogram $\mathcal{F}$ is a fundamental domain of the Λ-action on $\mathbb{C}$.) Let $z_1,\dots,z_r \in \mathbb{C}$ be the zeroes of f in $\mathcal{F}$ with multiplicities $m_1,\dots,m_r$. Let $w_1,\dots,w_s \in \mathbb{C}$ be the poles of f in $\mathcal{F}$ with multiplicities $n_1,\dots,n_s$. Then we have the following facts:

(i) The number of zeroes of f in $\mathcal{F}$ with multiplicities coincides with the number of poles of f in $\mathcal{F}$:

$$N = \sum_{i=1}^{r} m_i = \sum_{j=1}^{s} n_j.$$

The number N is called the *order* of the elliptic function f.

(ii) The positions of zeroes and poles of f in $\mathcal{F}$ with multiplicities are related as follows:

$$\sum_{i=1}^{r} m_i z_i - \sum_{j=1}^{s} n_j w_j \in \Lambda.$$

In case of the Weierstraß $\wp$-function, we have an even elliptic function (i.e., $\wp(-z) = \wp(z)$) of order 2, and with a pole of order 2 at $z = 0$. Its derivative $\wp'$ is an odd elliptic function (i.e., $\wp(-z) = -\wp(z)$) of order 3 with a pole of order 3 at $z = 0$.

Now we return to the above mentioned parametrisation: We learnt in the lectures that $T_\Lambda = \mathbb{C}/\Lambda$ is a Riemann surface, namely a real 2-dimensional torus. T_Λ has also a natural group structure inherited from the addition of complex numbers. It can also be shown that for any pair $(g_2, g_3) \in \mathbb{C}^2$ with $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice $\Lambda \subset \mathbb{C}$ such that $(g_2, g_3) = (g_2(\Lambda), g_3(\Lambda))$, where $g_2(\Lambda)$ and $g_3(\Lambda)$ was given above in (1) and (2). The parametrisation of C_F is $\Phi : T_\Lambda \to C_F$, given by

$$\Phi(z + \Lambda) = \begin{cases} [\wp_\Lambda(z), \wp'_\Lambda(z), 1] & \text{if } z \notin \Lambda, \\ [0, 1, 0] & \text{if } z \in \Lambda. \end{cases}$$

Below, you will show that Φ is bijective. It can be shown that Φ is even biholomorphic. The aim of this exercise is to show that

$$\Phi(z_1 + z_2 + \Lambda) = \Phi(z_1 + \Lambda) + \Phi(z_2 + \Lambda), \tag{3}$$

where "+" on the right hand side is addition in the cubic with $\mathcal{O} = [0, 1, 0] \in C_F$ as the identity element. The relation (3) explains why the geometric construction $P + Q = \mathcal{O} * (P * Q)$ defines, indeed, a commutative group structure on C_F. In short, (3) shows that Φ is also an isomorphism between abelian groups. (As an aside, this approach also provides an alternative proof of the associativity law in C_F.)

Now we are ready to formulate your tasks. Check the following facts:

(a) The point $\mathcal{O} = [0, 1, 0] \in \mathbb{P}^2_{\mathbb{C}}$ is a flex of C_F. Moreover, the tangent line of C_F at $\mathcal{O}$ is given by $L_0 : Z = 0$. Therefore, the only intersection point between L_0 and C_F is $\mathcal{O}$ (with intersection multiplicity 3).

(b) Φ maps points of T_Λ to points of C_F and Φ is injective (here you may use the facts (i) and (ii) above). (In fact, since images of compact sets under continuous maps are compact, and since holomorphic maps are open maps by the Open Mapping Theorem in Complex Analysis, the image $\Phi(T_\Lambda)$ is both open and closed in C_F. This implies that the map $\Phi : T_\Lambda \to C_F$ is also surjective, since C_F is connected.)

(c) Any point $P \in C_F \backslash \{\mathcal{O}\}$ is of the form $P = [a, b, 1]$, and we have $\mathcal{O} * P = [a, -b, 1]$.

(d) Any projective line $L \neq L_0$ is of the form $L : \alpha X + \beta Y + \gamma Z = 0$ with $(\alpha, \beta) \in \mathbb{C} \backslash \{(0, 0)\}$. Assume that $L \cap C_F$ has three distinct intersection points $P_1, P_2, P_3 \in \mathbb{P}^2_{\mathbb{C}}$. Show that the three points $z_1, z_2, z_3 \in \mathcal{F}$ with $P_j = \Phi(z_j + \Lambda)$ for $j = 1, 2, 3$ satisfy

$$z_1 + z_2 + z_3 \in \Lambda.$$

Hint: You may distinguish the two cases

- all three points $P_1, P_2, P_3 \in C_F$ differ from $\mathcal{O}$,
- one of the three points $P_j \in C_F$ agrees with $\mathcal{O}$,

and use the fact that $\alpha\wp(z) + \beta\wp'(z) + \gamma$ is an elliptic function with only pole in $\mathcal{F}$ at the origin.

(e) Prove (3) for any pair $z_1, z_2 \in \mathcal{F}$ with $z_1 \neq z_2$ and $z_1, z_2 \neq 0$. The other cases for z_1 and z_2 can be proved as well, but we do not ask for these proofs here.