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# Algebraic Geometry III/IV

**Problems, set 6.** To be handed in on **Wednesday, 5 March 2014**, in the lecture.

**Exercise 9.** This exercise is devoted to the derivation of the *Weierstraß normal form* of a cubic. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a non-singular cubic defined by the polynomial $F \in \mathbb{C}[X, Y, Z]$. We start as in last term's lectures (when we transformed $C$ into $C_F$ with $F(X, Y, Z) = Y^2Z - X(X-Z)(X-\lambda Z)$ with $\lambda \in \mathbb{C}\backslash\{0, 1\}$), and can assume that, after a suitable projective transformation, $P = [0, 1, 0] \in C_F$ is a flex and that $Z = 0$ is a tangent line to $C_F$ at $[0, 1, 0]$. Analogously as in last term's lectures, this implies that $F(X, Y, Z)$ has the form

$$F(X, Y, Z) = (\alpha X + \beta Y + \gamma Z)YZ + G(X, Z),$$

where $G(X, Z)$ is homogeneous of degree 3 and $\beta \neq 0$. Moreover, $G(X, Z)$ must contain a non-zero term $aX^3$ for, otherwise, $Z$ would be factor of $F(X, Y, Z)$ and $C_F$ would be reducible and, therefore, singular. You don't need to prove this first step again. Therefore, we can start with the form

$$F(X, Y, Z) = aX^3 + bX^2Z + cXYZ + dXZ^2 + eY^2Z + fYZ^2 + gZ^3,$$

with $a \neq 0$ and $e \neq 0$.

(a) Show that the substitution of $Y$ by $Y - \frac{c}{2e}X - \frac{f}{2e}Z$ implies vanishing of the coefficients of $XYZ$ and $YZ^2$, and that no new non-zero terms are generated. So, another projective transformation yields

$$F(X, Y, Z) = a'X^3 + b'X^2Z + d'XZ^2 + e'Y^2Z + gZ^3,$$

still with $a', e' \neq 0$.

(b) Show that substitution of $X$ by $X - \frac{b'}{3a'}Z$ yields the equation

$$F(X, Y, Z) = a''X^3 + d''XZ^2 + e''Y^2Z + g''Z^3,$$

still with $a'', e'' \neq 0$.

(c) Argue, why we can, after another projective transformation, obtain the final equation

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3, \tag{1}$$

for the cubic $C$.

(d) Show that (1) defines a non-singular cubic if and only if $g_2^3 - 27g_3^2 \neq 0$.

**Additional remarks to this exercise:** The function $j = \frac{g_2^3}{g_2^3 - 27g_3^2}$ turns out to be a projective invariant of the Weierstraß normal form. Two normal forms $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$ of non-singular cubics are projectively equivalent if and only if the corresponding values of $j$ coincide. In particular, there are uncountably many projectively non-equivalent non-singular cubics. The final classification of all cubics (non-singular and singular) looks as follows:

(i) Every non-singular cubic is projectively equivalent to a curve of the type $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$.

(ii) Every irreducible singular cubic is projectively equivalent to the curve $X^3 + Y^3 - XYZ = 0$ (cubic with a nodal singularity) or to the curve $X^3 - Y^2Z = 0$ (cubic with a cuspidal singularity).

(iii) Every reducible cubic $C$ is either a conic plus a chord, a conic plus a tangent line, or $C$ consists of three lines $L_1, L_2, L$ which meet in three different points (triangle), in one common point (triple point), or two or three of the lines $L_j$ coincide.