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Algebraic Geometry III/IV

Problems, set 9.

Exercise 12. Let $\mathcal{L}$ be the set of all projective lines in $\mathbb{P}^2_{\mathbb{C}}$, i.e.,

$$\mathcal{L} = \{C_{uX+vY+wZ} \mid (u,v,w) \neq 0\}.$$

There is a natural identification $\Phi : \mathcal{L} \to \mathbb{P}^2_{\mathbb{C}}$, given by $C_{uX+vY+wZ} \mapsto [u,v,w]$. This identification leads to the definition of the dual of a projective algebraic curve: Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ is a non-singular algebraic curve and $\mathcal{T}(C)$ be the set of all tangent lines of C. Then the **dual** of C is the set $C^* = \Phi(\mathcal{T}(C)) \subset \mathbb{P}^2_{\mathbb{C}}$. Prove the following facts:

(a) Let

$$A = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

A projective conic $C_F \subset \mathbb{P}^2_{\mathbb{C}}$, given by the equation

$$F(X,Y,Z) = \begin{pmatrix} X & Y & Z \end{pmatrix} A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

is non-singular if and only if $\det A \neq 0$.

(b) Let $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ be a non-singular conic and F as in (a). Let $P = [\alpha, \beta, \gamma] \in C_F$. Then the tangent line of C_F at P is given by the equation

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0.$$

(c) Let $C = C_F$ be a non-singular conic and F as in (a). Then the dual $C^* \subset \mathbb{P}^2_{\mathbb{C}}$ of the conic C is again a non-singular conic $C^* = C_G$, given by the equation

$$G(X, Y, Z) = \begin{pmatrix} X & Y & Z \end{pmatrix} A^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

Exercise 13. Find the genus of a non-singular model of the irreducible curve $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ with $F(X, Y, Z) = 3Y^4 + 4Y^3Z + X^4$.

Exercise 14. Find the genus of a non-singular model of the irreducible curve $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ with $F(X, Y, Z) = Y^4 - 2X^2Y^2 + XZ^3$.

Exercise 15. Find the genus of a non-singular model of the irreducible curve $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ with $F(X, Y, Z) = X^5 + 3Y^5 - 5Y^3Z^2$.