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Algebraic Geometry III/IV

Solutions, set 4.

Exercise 6.

(a) For $p = (x, y, z) \in S^2$, we have

$$L_{np} = \{t(x,y,z)+(1-t)(0,0,1) \mid t \in \mathbb{R}\} = \{(tx, ty, 1+t(z-1)) \mid t \in \mathbb{R}\}.$$

The intersection $L_{np} \cap V_1$ is calculated by equating $1 + t(z - 1) = 0$, i.e., $t = 1/(1 - z)$. This leads to

$$\phi_1(x, y, z) = \frac{x}{1-z} + \frac{y}{1-z}i.$$

Analogously, we obtain

$$\phi_2(x, y, z) = \frac{x}{1+z} - \frac{y}{1+z}i.$$

(b) We identify $z = u + vi \in \mathbb{C}$ with $p_0 = (u, v, 0) \in \mathbb{R}^3$ and obtain

$$L_{np_0} = \{(tu, tv, 1 - t) \mid t \in \mathbb{R}\}.$$

The intersection $L_{np_0} \cap S^2$ is calculated via

$$(tu)^2 + (tv)^2 + (1 - t)^2 = 1,$$

i.e.,

$$t^2(u^2 + v^2 + 1) = 2t.$$

Solutions are then $t = 0$ (corresponding to the point $n \in S^2$) and $t = \frac{2}{1+u^2+v^2} = \frac{2}{1+|z|^2}$ (corresponding to the point $\phi_1^{-1}(z) \in S^2$). We obtain

$$\phi_1^{-1}(z) = \left(\frac{2\mathrm{Re}(z)}{1+|z|^2}, \frac{2\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right).$$

Analogously, identifying $z = u + vi \in \mathbb{C}$ with $p_0 = (u, -v, 0 \in \mathbb{R}^3$ we obtain

$$\phi_2^{-1}(z) = \left(\frac{2\mathrm{Re}(z)}{1+|z|^2}, -\frac{2\mathrm{Im}(z)}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right).$$

(c) We first check that

$$\phi_1(S^2\backslash\{n,s\}) = \phi_2(S^2\backslash\{n,s\}) = \mathbb{C}\backslash\{0\}.$$

Therefore, we have $\phi_2\phi_1^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$. Moreover, we obtain for $z \in \mathbb{C}\backslash\{0\}$

$$\begin{aligned}\phi_2 \circ \phi_1^{-1}(z) &= \phi_2\left(\frac{2\mathrm{Re}(z)}{1+|z|^2}, \frac{2\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right) = \phi_2(X,Y,Z) \\ &= \frac{X}{1+Z} - \frac{Y}{1+Z}i.\end{aligned}$$

We have $1+Z = \frac{2|z|^2}{1+|z|^2}$ and, therefore,

$$\phi_2 \circ \phi_1^{-1}(z) = \frac{1+|z|^2}{2|z|^2}\frac{2\mathrm{Re}(z)}{1+|z|^2} - \frac{1+|z|^2}{2|z|^2}\frac{2\mathrm{Im}(z)}{1+|z|^2}i = \frac{\bar{z}}{|z|^2} = \frac{1}{z}.$$

Analogously, we obtain

$$\phi_1 \circ \phi_2^{-1}(z) = \frac{1+|z|^2}{2|z|^2}\frac{2\mathrm{Re}(z)}{1+|z|^2} - \frac{1+|z|^2}{2|z|^2}\frac{2\mathrm{Im}(z)}{1+|z|^2}i = \frac{\bar{z}}{|z|^2} = \frac{1}{z}.$$

In both cases, the coordinate changes are holomorphic functions, finishing the proof that S^2 is a Riemann surface.

Exercise 7.

(a) We choose $f : \mathbb{P}^1_{\mathbb{C}} \to S^2$ as follows:

$$f([a,b]) = \begin{cases}\phi_1^{-1}(a/b) & \text{if } b \neq 0, \\ \phi_2^{-1}(b/a) & \text{if } a \neq 0.\end{cases}$$

we first have to check whether this map is well defined, i.e., whether $\phi_1^{-1}(1/z) = \phi_2(z)$ for all $z \neq 0$:

$$\begin{aligned}\phi_1^{-1}(1/z) &= \left(\frac{2\mathrm{Re}(1/z)}{1+|1/z|^2}, \frac{2\mathrm{Im}(1/z)}{1+|1/z|^2}, \frac{|1/z|^2-1}{1+|1/z|^2}\right) \\ &= \left(\frac{2\mathrm{Re}(\bar{z})}{|z|^2+1}, \frac{2\mathrm{Im}(\bar{z})}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right) \\ &= \left(\frac{2\mathrm{Re}(z)}{1+|z|^2}, -\frac{2\mathrm{Im}(z)}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right) \\ &= \phi_2^{-1}(z).\end{aligned}$$

We recall from the lectures that $\mathbb{P}^1_{\mathbb{C}}$ is a Riemann surface via the following coordinate charts $\psi_1 : \{[a,b] \mid b \neq 0\} \to \mathbb{C}$, $\psi_1([a,b]) = a/b$, and $\psi_2 : \{[a,b] \mid a \neq 0\} \to \mathbb{C}$, $\psi_2([a,b]) = b/a$. Then we have

$$\phi_1 \circ f \circ \psi_1^{-1}(z) = \phi_1 \circ f([z,1]) = \phi_1 \circ \phi_1^{-1}(z/1) = z,$$

and

$$\phi_2 \circ f \circ \psi_2^{-1}(z) = \phi_2 \circ f([1,z]) = \phi_2 \circ \phi_2^{-1}(z/1) = z,$$

i.e., both compositions are holomorphic. Similarly, we obtain

$$\phi_1 \circ f \circ \psi_2^{-1}(z) = \frac{1}{z}, \quad \phi_2 \circ f \circ \psi_1^{-1}(z) = \frac{1}{z},$$

as maps $\mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$. So all maps $\phi_j \circ f \circ \psi_i^{-1}$ are holomorphic and, therefore, f is a holomorphic map. One checks that the inverse map $f^{-1} : S^2 \to \mathbb{P}^1_{\mathbb{C}}$ is given by

$$f^{-1}(x,y,z) = \begin{cases} [\phi_1(x,y,z), 1] & \text{if } (x,y,z) \neq n, \\ [1, \phi_2(x,y,z)] & \text{if } (x,y,z) \neq s. \end{cases}$$

Since all compositions $\phi_j \circ f \circ \psi_i^{-1}$ are even biholomorphic and

$$\left(\phi_j \circ f \circ \psi_i^{-1}\right)^{-1} = \psi_i \circ f^{-1} \circ \phi_j^{-1},$$

we conclude that f^{-1} is also holomorphic.

(b) We first check that $g([a,b]) \in C_F$:

$$F(ab, a^2, b^2) = a^2b^2 - a^2b^2 = 0.$$

Next, we check that g is **bijective** by giving a formula for g^{-1}:

$$g^{-1}([x,y,z]) = \begin{cases} [y,x], & \text{if } y \neq 0 \\ [x,z], & \text{if } z \neq 0 \end{cases}.$$

Note first that if $[x,y,z] \in C_F$ then we cannot have $y = z = 0$ since then we would also have $x^2 = yz = 0$, i.e., $x = 0$, which cannot be. In the case $[x,y,z] \in C_F$ and $y \neq 0$ and $z \neq 0$ we have

$$[y,x] = [yz, xz] = [x^2, xz] = [x,z],$$

so g^{-1} is well defined. We easily check for $[x, y, z] \in C_F$ and $y \neq 0$ that

$$g(g^{-1}([x, y, z])) = g([y, x]) = [yx, y^2, x^2] = [x, y, x^2/y] = [x, y, z]$$

and

$$g^{-1}(g([y, x])) = g^{-1}([yx, y^2, x^2]) = [y^2, yx] = [y, x].$$

Similarly, we obtain for $[x, y, z] \in C_F$ and $z \neq 0$

$$g(g^{-1}([x, y, z])) = g([x, z]) = [xz, x^2, z^2] = [x, x^2/z, z] = [x, y, z]$$

and

$$g^{-1}(g([x, z])) = g^{-1}([xz, x^2, z^2]) = [xz, z^2] = [x, z].$$

C_F can be covered by the following two coordinate charts:

$$U_1 = \{[a, b, c] \in C_F \mid b \neq 0\}, \quad U_2 = \{[a, b, c] \in C_F \mid c \neq 0\},$$

$V_1 = V_2 = \mathbb{C}$ and $\phi_1 : U_1 \to V_1$, $\phi_1([a, b, c]) = a/b$, and $\phi_2 : U_2 \to V_2$, $\phi_2([a, b, c]) = a/c$. Then we have

$$\phi_1^{-1}(z) = [z, 1, z^2], \quad \phi_2^{-1}(z) = [z, z^2, 1].$$

For **biholomorphicity** of g, we have to check again that all the compositions $\phi_j \circ g \circ \psi_i^{-1}$ and $\psi_i \circ g \circ \phi_j^{-1}$ are biholomorphic. (Here ψ_i are the coordinate charts from part (a).) We only consider the example $\phi_1 \circ g \circ \psi_1^{-1}$:

$$\phi_1 \circ g \circ \psi_1^{-1}(z) = \phi_1 \circ g([z, 1]) = \phi_1([z, z^2, 1]) = z/z^2 = 1/z,$$

which, as a map $\mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$, is biholomorphic.