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Algebraic Geometry III/IV

Solutions, set 5.

Exercise 8.

(a) Let $F(X,Y,Z) = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3$ with $g_2^3 - 27g_3^2 \neq 0$. Then

$$\begin{aligned} F_X &= -12X^2 + g_2Z^2, \\ F_Y &= 2YZ, \\ F_Z &= Y^2 + 2g_2XZ + 3g_3Z^2, \end{aligned}$$

and

$$F_{XX} = -24X, \quad F_{XY} = 0, \quad F_{XZ} = 2g_2Z,$$

and

$$F_{YY} = 2Z, \quad F_{YZ} = 2Y, \quad F_{ZZ} = 2g_2X + 6g_3Z.$$

Therefore,

$$\mathcal{H}_F(X,Y,Z) = \det \begin{pmatrix} -24X & 0 & 2g_2Z \\ 0 & 2Z & 2Y \\ 2g_2Z & 2Y & 2g_2X + 6g_3Z \end{pmatrix}$$

and

$$\mathcal{H}_F(0,1,0) = \det \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = 0.$$

Since $F(0,1,0) = 0$ and F is non-singular, we conclude that $\mathcal{O} = [0,1,0]$ is a flex of C_F. The tangent line L_0 of C_F at $\mathcal{O}$ is given by

$$XF_X(0,1,0) + YF_Y(0,1,0) + ZF_Z(0,1,0) = Z = 0.$$

It was mentioned in the lectures that the tangent line of C_F at a flex has only intersection point with C_F, namely, the flex itself, and that the intersection multiplicity at this point is 3.

(b) First of all, we see that $[0,1,0] \in C_F$ since $F(0,1,0) = 0 - 4 \cdot 0 + g_2 0 + g_3 0 = 0$. Next, we obtain

$$F(\wp(z), \wp'(z), 1) = \wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3 = 0,$$

employing the differential equation for the Weierstrass $\wp$-function. Next, we prove injectivity of Φ: Let $\Phi(z_1 + \Lambda) = \Phi(z_2 + \Lambda)$ for $z_1, z_2 \in \mathcal{F}$. We check easily that $z_1 = z_2$ in the case if one of z_1, z_2 is equal to 0. Now assume that $z_1 \neq 0 \neq z_2$ and $z_1 \neq z_2$. This implies that $\wp(z_1) = \wp(z_2) \neq \infty$ and $\wp'(z_1) = \wp'(z_2) \neq \infty$. Since $\wp$ is an elliptic function of order 2 and $z = 0$ is a pole of order 2, we conclude from (i) and (ii) that $z_1 + z_2 \in \Lambda$. Since $\wp'$ is an elliptic function of order 3 and $z = 0$ is a pole of order 3, we conclude from (i) and (ii) that there must exist $z_3 \in \mathbb{C}$ with $\wp'(z_3) = \wp'(z_1)$ and $z_1 + z_2 + z_3 \in \Lambda$. Both results together imply that $z_3 \in \Lambda$, but $\wp'$ has then a pole at z_3. This implies that $\infty \neq \wp'(z_1) = \wp'(z_3) = \infty$, and we have a contradiction.

(c) Let $P = [a,b,0] \in C_F$. Then we have $0 = F(a,b,0) = -4a^3$, i.e., $a = 0$ and $P = [0,1,0] = \mathcal{O}$. Therefore, any point $P \backslash \{\mathcal{O}\}$ is of th form $P = [a,b,1]$. The projective line through $\mathcal{O} = [0,1,0]$ and $P = [a,b,1] \in C_F$ is $L = \{[sa, sb+t, s] \mid (s,t) \in \mathbb{C}^2 \backslash 0\}$. Note that $P \in C_F$ means that

$$0 = F(a,b,1) = b^2 - 4a^3 - g_2 a - g_3.$$

The third point of intersection between C_F and L is then $[a,-b,1] \in L$, since

$$F(a,-b,1) = (-b)^2 - 4a^3 - g_2 a - g_3 = F(a,b,1) = 0.$$

This shows that

$$\mathcal{O} * P = [a,-b,1].$$

(d) If $(\alpha, \beta) = (0,0)$, the projective line L would be given by $L : Z = 0$ and, therefore, coincide with L_0. Therefore, $(\alpha, \beta) \neq (0,0)$.

Assume first that $\beta \neq 0$ and $\mathcal{O} = [0,1,0]$ cannot lie on L. Therefore, all three distinct intersection points P_1, P_2, P_3 differ from $\mathcal{O}$. Since the points on $C_F \backslash \{\mathcal{O}\}$ are of the form $[\wp(z), \wp'(z), 1]$, the preimages $z_j \in mathcal F$ of the intersection points P_j with L must satisfy $\alpha\wp(z) + \beta\wp'(z) + \gamma = 0$. Let $g = \alpha\wp + \beta\wp' + \gamma$. Since $\beta \neq 0$, g is an elliptic function of order 3 with only pole in $\mathcal{F}$ at the origin and of order 3. As mentioned before, the three points $z_1, z_2, z_3 \in \mathcal{F}$ with $\Phi(z_j + \Lambda) = P_j$ for $j = 1, 2, 3$ must satisfy $g(z_j) = 0$. We conclude from (ii) on the exercise sheet that $z_1 + z_2 + z_3 \in \Lambda$.

Assume next that $\beta = 0$. Then $\alpha \neq 0$ and $\mathcal{O} = [0, 1, 0]$ is one of the three distinct intersection points. W.l.o.g., we assume that $P_1 = \mathcal{O}$. Then $P_1 = \Phi(0 + \Lambda$. This means that the preimage of P_1 is $z_1 = 0 \in \mathcal{F}$. As before, the preimages z_2, z_3 of the remaining two intersection points P_2, P_3 with L must satisfy $g(z_j) = 0$ with $g = \alpha\wp + \gamma$. g is now an elliptic function of order 2 with only pole in $\mathcal{F}$ at the origin and of order 2. Again, we conclude from (ii) on the exercise sheet that $z_2 + z + 3 \in \Lambda$ which implies, since $z_1 = 0$, also $z_1 + z_2 + z_3 \in \Lambda$.

(e) Let $P_1 = \Phi(z_1 + \Lambda)$ and $P_2 = \Phi(z_2 + \Lambda)$. Let $P_3 = P_1 * P_2$ and $z_3 \in \mathcal{F}$ with $\Phi(z_3 + \Lambda) = P_3$. We conclude from (c) that $z_1 + z_2 + z_3 \in \Lambda$. We distinguish two cases.

Assume first that $z_3 = 0$, i.e., $P_3 = \mathcal{O}$. Since $\mathcal{O}$ is a flex of C_F, we conclude that $\mathcal{O} = P_1 + P_2 = \Phi(z_1 + \Lambda) + \Phi(z_2 + \Lambda)$. On the other hand we conclude from $z_3 = 0$ and $z_1 + z_2 + z_3 \in \Lambda$ that $z_1 + z_2 \in \Lambda$ and, therefore, $\Phi(z_1 + z_2 + \Lambda) = \Phi(0 + \Lambda) = \mathcal{O}$, proving (3) in this case.

Assume next that $z_3 \neq 0$, i.e., $P_3 \neq \mathcal{O}$. We conclude from $z_1 + z_2 + z_3 \in \Lambda$ and $-(z_1 + z_2) \notin \Lambda$ (because of $z_3 \neq 0$) that

$$\begin{aligned} P_1 * P_2 = P_3 &= \Phi(-(z_1 + z_2) + \Lambda) \\ &= [\wp(-(z_1 + z_2)), \wp'(-(z_1 + z_2)), 1] \\ &= [\wp(z_1 + z_2), -\wp'(z_1 + z_2), 1]. \end{aligned}$$

We conclude from (b) that

$$P_1 + P_2 = \mathcal{O} * P_3 = [\wp(z_1 + z_2), \wp'(z_1 + z_2), 1] = \Phi(z_1 + z_2 + \Lambda),$$

proving (3) in this case.