Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 11 18.1.2012

Do **Exercises 2 and 4** as **homework for this week**. The cumulative homework over the coming weeks will be collected and marked in a few weeks time. Try also to do Exercise 5, even though this exercise will not be marked. Have a look at all solutions when you receive the solution sheet the following week.

1. This is a little warmup exercise about exterior derivatives, wedge products and pullbacks.

   (a) Let ω_1 and ω_2 be two differential forms on $U \subset \mathbb{R}^n$. Assume that ω_1 is closed and ω_2 is exact. Show that $\omega_1 \wedge \omega_2$ is exact.

   (b) Let $U = \mathbb{R}^2 \times (0, \infty)$ and $\varphi : U \to U$,

   $$\varphi(x_1, x_2, x_3) = (y_1, y_2, y_3) = (e^{x_3} x_1, e^{-x_3} x_2, x_3^2).$$

   Let $\omega = y_1^2 y_2 dy_1 \wedge dy_3 \in \Omega^2(U)$. Calculate $d\omega$ and $\varphi^*(d\omega)$. Then calculate $\varphi^*(\omega)$ and $d\varphi^*(\omega)$. (If you didn't make a mistake, you should have $\varphi^*(d\omega) = d\varphi^*(\omega)$.)

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

   $$f(x_1, x_2, x_3) = (y_1, y_2, y_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3).$$

   (a) Calculate the pullback $\omega = f^*(y_3 dy_1 \wedge dy_2 \wedge dy_3)$.

   (b) Calculate $\int_{(1,2)\times(0,2\pi)\times(0,1)} \omega$.

3. Prove the *Transformation Rule* in the following special case: Let $U = \times_{i=1}^n (a_i, b_i)$ and $V = \times_{i=1}^n (c_i, d_i)$ and $\varphi : U \to V$ a diffeomorphism of the form

   $$\varphi(x_1, \ldots, x_n) = (\varphi_1(x_1), \ldots, \varphi_n(x_n)).$$

   Let $f : V \to \mathbb{R}$ be a bounded integrable function. Using Fubini and the one-dimensional Substitution Rule for integrals, show that

   $$\int_V f(y) dy = \int_U f \circ \varphi(x) |\det D\varphi(x)| dx.$$

4. Let $A_1, A_2, \ldots$ be a countable sequence of set of measure zero in $\mathbb{R}^n$. Show that the union $\bigcup_{i=1}^\infty A_i$ is, again, a set of measure zero. Carefully justify all your arguments. In particular, when giving a covering of the union, explain in detail why the sets in this covering are countably many.

5. Let $\omega = \frac{dx \wedge dy}{y^2}$ be the volume form of the hyperbolic upper half plane $\mathbb{H}^2$. Check the following facts:

(a) Let $f : \mathbb{H}^2 \to \mathbb{H}^2$, $f(z) = z + b$, $b \in \mathbb{R}$. Show that $f^*\omega = \omega$.

(b) Let $g : \mathbb{H}^2 \to \mathbb{H}^2$, $g(z) = az$, $a > 0$. Show that $g^*\omega = \omega$.

(c) Let $h : \mathbb{H}^2 \to \mathbb{H}^2$, $h(z) = 1/z$. Show that $h^*\omega = \omega$.

Since the maps f, g, h generate the Möbius transforms $k : \mathbb{H}^2 \to \mathbb{H}^2$, $k(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, we conclude from the above calculations that the Möbius transforms preserve the volume form ω of the hyperbolic upper half plane, i.e., the ω-volume (= hyperbolic area) of a set is preserved under Möbius transforms.

Hint: Write the functions f, g, h first as maps $\mathbb{R} \times (0, \infty) \to \mathbb{R} \times (0, \infty)$, before you start your calculations.