Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 14 8.2.2012

Do **Exercises 2 and 5** as **homework for this week**. These homework exercises will not be marked, but you can check your solutions against the solution sheet in the following week.

1. (Easy Warmup!) Determine the critical points of

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (x^2, 2x + e^x \cos(y), xy \sin(xy))$.

   (b) $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by $g(x, y, z) = (2x^2 + (y - 1)^2, z(\cos(y) - 1))$.

2. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^2 + 2z^2 = 4\}$.

   (a) Show that M is a manifold.

   (b) For $p = (-1, 1, 1)$, determine the tangent space T_pM.

3. Let $A : (-\epsilon, \epsilon) \to GL(n, \mathbb{R})$ be a smooth curve.

   (a) Prove that

   $$(\det A)'(t) = (\det A(t))\mathrm{tr}(A(t)^{-1}A'(t)).$$

   Hint: Let $a_1(t), \ldots, a_n(t)$ denote the columns of $A(t)$. You may use the product rule for n factors to conclude that,

   $$(\det A)'(t) = \sum_{j=1}^{n} \det(a_1(t) \ldots a_j'(t) \ldots a_n(t)).$$

   Use the fact that $a_1(t), \ldots, a_n(t)$ form a basis of $\mathbb{R}^n$ to write $a_j'(t)$ in terms of $a_1(t), \ldots, a_n(t)$, i.e., $A'(t) = A(t) \cdot (\alpha_{ij}(t))$, and conclude that

   $$(\det A)'(t) = \det A(t) \cdot \mathrm{tr}(\alpha_{ij}(t)).$$

   (b) Use (a) and Exercise 4, Sheet 13, to show that

   $$T_{\mathrm{Id}}SL(n, \mathbb{R}) = \{B \in M(n, \mathbb{R}) \mid \mathrm{tr}\, B = 0\},$$

   where Id is the identity matrix in $SL(n, \mathbb{R})$.

4. Show that

   $$M = \{(x, y, z) \in \mathbb{R}^3 \mid (x - 1)^2 + y^2 = 5, y = z\}$$

   is a compact manifold and the extremal values of $f(x, y, z) = x^2 + y^2 + z$ on M are 11 and 1.

5. (a) Find the point of the sphere $x^2 + y^2 + z^2 = 1$ which is at the greatest distance from the point $(1, 2, 3) \in \mathbb{R}^3$.

(b) Find the rectangle of greatest perimeter inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \quad = \quad 1.$$

6. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that

$$1 \quad \leq \quad \frac{1}{p}\, u^p + \frac{1}{q}\, v^q$$

for all positive numbers u, v with $u \cdot v = 1$.

Hint: Lagrange multipliers.

(b) Show that

$$uv \quad \leq \quad \frac{1}{p}\, u^p + \frac{1}{q}\, v^q$$

for all $u, v \geq 0$.

Remark: Note that this exercise provides an alternative proof of (1) from Exercise 5, Sheet 5.