Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 16 29.2.2012

Do **Exercise 1** as **homework for this week**. This homework exercise will not be marked, but you can check your solution against the solution sheet in the following week.

1. For $a, b, c > 0$ consider the ellipsoid

$$E := \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}.$$

Let ω be the following differential form on $\mathbb{R}^3$;

$$\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

   (a) Calculate $d\omega$.

   (b) Find an almost global parametrisation of E such that the outward unit normal vector field is positively oriented. Calculate

$$\int_E \omega.$$

   Hint: Think of polar coordinates on the sphere.

2. Let $U \subset \mathbb{R}^n$ be open and starlike and $\omega \in \Omega^k(U)$, $k \geq 1$ with $d\omega = 0$. The aim of this exercise is to prove **Poincaré's Lemma** in its general form, i.e., that there is an $\alpha \in \Omega^{k-1}(U)$ with $d\alpha = \omega$.

Henceforth we will denote the coordinate functions of $\mathbb{R} \times U$ by $t, x_1, \ldots, x_n$. Note that every differential form $\eta \in \Omega^k(\mathbb{R} \times U)$ is then of the form

$$\eta = \eta_1 + dt \wedge \eta_2, \tag{1}$$

where

$$\eta_1 = \sum_{i_1 < \cdots < i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and

$$\eta_2 = \sum_{j_1 < \cdots < j_{k-1}} g_{j_1,\ldots,j_{k-1}} dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}},$$

with $f_{i_1,\ldots,i_k}, g_{j_1,\ldots,j_{k-1}} \in C^\infty(\mathbb{R} \times U)$.

Since U is starlike, there is a point $p \in U$ and a map $H : \mathbb{R} \times U \to \mathbb{R}^n$, defined by $H(t, x) = p + t(x - p)$, such that $H(t, x) \in U$ for all $t \in [0, 1]$ and $x \in U$ (since $H([0, 1], x)$ is the straight line segment from p to x). Observe that $H(0, x) = p$ and $H(1, x) = x$. Let $i_t : U \to \mathbb{R} \times U$ be the inclusion of U into $\mathbb{R} \times U$ at "level" t, i.e., $i_t(x) = (t, x)$.

Finally, let $I : \Omega^k(\mathbb{R} \times U) \to \Omega^{k-1}(U)$ be defined by

$$(I\eta)_x(v_1, \ldots, v_{k-1}) = \int_0^1 \eta_2(t,x)(Di_t(x)(v_1), \ldots, Di_t(x)(v_k)) \, dt,$$

if $\eta = \eta_1 + dt \wedge \eta_2$ as given in (1).

(a) Prove that $i_1^*\eta - i_0^*\eta = d(I\eta) + I(d\eta)$.

(b) Using (a) and $H \circ i_1 = \text{id}$ and $H \circ i_0 = \text{constant}$, show that

$$\omega = d\alpha$$

with $\alpha = I(H^*\omega)$.

Hint: Note that if $F = \text{constant}$, then $DF(x) = 0$ for all x, and therefore $F^*\omega = 0$.