Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 3 24.10.2011

Do **Exercise 2** as **homework for this week**. The cumulative homework over the coming weeks will be collected and marked in a few weeks time. Try to do at least one of the other exercises as well for your own benefit. All of them are very useful. Have a look at all solutions when you receive the solution sheet the following week.

1. Let (M, d_M) and (N, d_N) be metric spaces and $f : M \to N$ be a continuous function. Show the following facts:

   (a) We have $f(x_n) \to f(x)$ for all convergent sequences $x_n \to x$.

   (b) If $K \subset M$ is compact then so is $f(K) \subset N$.

   (c) If $U \subset N$ is an open set then so is $f^{-1}(U) \subset M$. (In fact, this property is even equivalent to continuity of f; you don't need to prove this equivalence!)

   (d) If $A \subset N$ is a closed set then so is $f^{-1}(A) \subset M$.

   Hint: The explanations in the previous Problem Class might be helpful.

2. Let $f : M \to N$ be a continuous map between metric spaces. Show: If M is compact, then f is uniformly continuous.

   Hint: Negate the fact that f is uniformly continuous. Then construct a sequence in M with a certain property. Use (sequential compactness) of M to obtain a contradiction to the continuity property at a limit point.

3. Let $f : M \to \mathbb{R}$ be a continuous function and (M, d) be a compact metric space. Show that f assumes its infimum and supremum, i.e., has a well defined finite minimum and maximum.

   Hint: Use sequential compactness.

4. Let $\|\cdot\|$ be the norm of the inner product space $(V, \langle\cdot,\cdot\rangle)$. Show that the parallelogram equation is satisfied:

   $$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

   Let $C([a, b])$ be equipped with the supremum norm. Show that the parallelogram equation is not satisfied and, hence, the supremum norm is not induced from an inner product space,

5. Show that all norms in $\mathbb{R}^n$ are equivalent.

   Hint: You only need to show that every norm $\|\cdot\|$ in $\mathbb{R}^n$ is equivalent to the Euclidean norm $\|\cdot\|_2$. Show first that $\|\cdot\| \leq C\|\cdot\|_2$ for a suitable constant $C > 0$ (argue with vectors $v = \sum a_i e_i$, where e_i is the standard basis of $\mathbb{R}^n$). Conclude from this that the map $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ is continuous with respect to the Euclidean metric. Then the map

$$\|\cdot\| : S^{n-1} \to [0,\infty)$$

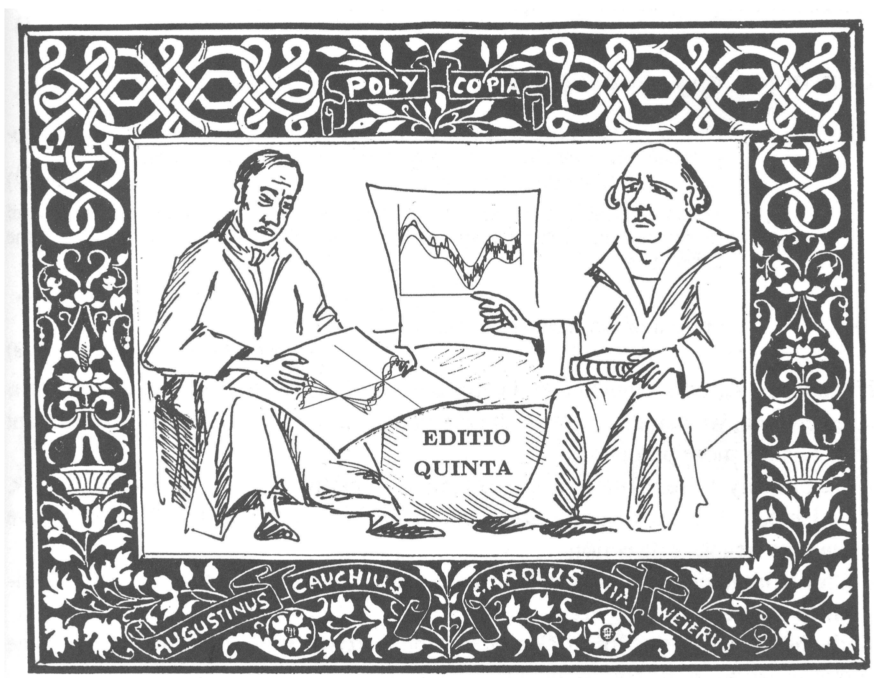
(with $S^{n-1} \subset \mathbb{R}^n$ the Euclidean unit sphere) assumes its minimum and maximum. Use this fact to conclude the equivalence.

For your amusement:

The following illustration is taken from the book "E. Hairer/G. Wanner: Analysis by its history". The picture on Weierstraß' paper should look familiar to you. Abel's counterexample can be found in the Oevres (1826), vol. 1, p. 224-225: *The following theorem can be found in the work of Mr. Cauchy: "If the various terms of the series* $u_0 + u_1 + u_2 + \ldots$ *are continuous functions,... then the sum s of the series is also a continuous function of x." But it seems to me that this theorem admits exceptions. For example the series*

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \ldots$$

is discontinuous at each value $(2m+1)\pi$ *of x,...*



Weierstrass explains uniform convergence to Cauchy
who meditates over Abel's counterexample
(Drawing by K. Wanner)