Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 6 16.11.2011

Do **Exercises 3, 5 and 6** as **homework for this week**. These homework exercises will not be marked, but you can check your solutions against the solution sheet in the following week. It is really important that you do every week the emphasized questions in order to stay up to date with the course.

1. (Easy Warmup) Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = Ax + b$ with $A \in M_{n,n}(\mathbb{R})$ and $b \in \mathbb{R}^n$. Show that $Df(x) = A$ for all $x \in \mathbb{R}^n$. Let $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(x) = \langle x, Ax \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Show that $Dg(x) = x^\top (A + A^\top)$.

2. **True or False?** Let $c : [a, b] \to \mathbb{R}^2$ be a smooth curve and $c_n : [a, b] \to \mathbb{R}^2$ a sequence of smooth curves with
$$\max_{t \in [a,b]} \|c(t) - c_n(t)\|_2 \to 0 \qquad \text{as } n \to \infty.$$
Does this imply $L(c_n) \to L(c)$?

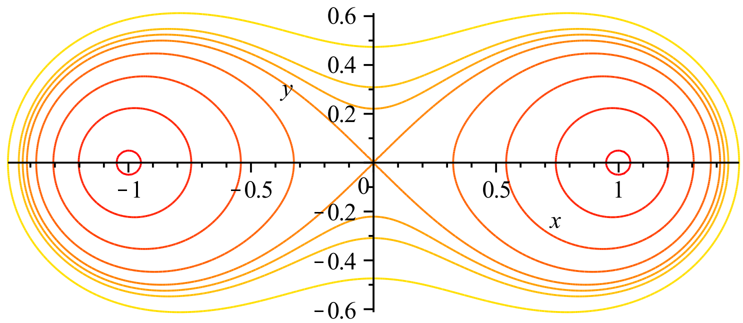
   Hint: Approximate a diagonal straight line by horizontal and vertical line segments.

3. (a) Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be differentiable with $c = f(x_0)$. Let $\gamma : [a, b] \to U_c$ be a differentiable curve in the level set $U_c = \{x \in U \mid f(x) = c\}$. Show that
$$\langle \gamma'(t), \nabla f(\gamma(t)) \rangle = 0,$$
where $\langle \cdot, \cdot \rangle$ denotes the standard inner product.

   (b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function satisfying
$$\nabla g(x) = h(x) \cdot x,$$
where $h : \mathbb{R}^2 \to \mathbb{R}$. Show that g is constant on the circles $\{x \in \mathbb{R}^2 \mid \|x\| = r\}$ with $r > 0$.

4. The *cycloid* is given by $c : \mathbb{R} \to \mathbb{R}^2$,
$$c(t) = (rt - r\sin(t), r - r\cos(t)),$$
and is the trace of a peripheral point of a disk of radius $r > 0$ rolling on a horizontal plane. Calculate the length of a full turn of the cycloid.

5. Let $c > 0$ and $C_c \subset \mathbb{R}^2$ be the set of all points for which the product of the distances to the focal points $(-1, 0)$ and $(1, 0)$ is equal to c^2. Show that C_c is the set of solutions of
$$F(x, y) = (x^2 + y^2)^2 - 2x^2 + 2y^2 = c^4 - 1.$$
These curves are named after the Italian/French astronomer Giovanni Domenico Cassini, who introduced them in connection with "the movements of the sun and its distances to the Earth" in 1680.



A special case is $c = 1$, the so-called *lemniscate* of JACOB BERNOULLI (1654-1705). Find the set of points with $\frac{\partial F}{\partial y} = 0$, i.e., the points where the Implicit Function Theorem does not apply (and explain this with the help of the above picture). Find also the set of points (x, y) with $x \neq 0$ and $y'(x) = 0$, i.e., when the *cassinian curves* (level sets of F) assume extremal y-values and show that they lie on a circle.

6. Let $f : (-1/2, \infty) \times \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x_1, x_2) = (x_1^2, x_1 + x_2^3).$$

Is f locally invertible at $x = (0, 0)$? Give a reason why f is locally invertible at $x = (1, 1)$ and determine $Df^{-1}(1, 2)$.

7. Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^n$ be a differentiable vector field with component functions $F_1, \ldots, F_n$. Then $\operatorname{div} F(x) = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(x)$. The Laplacian of a twice differentiable function $f : U \to \mathbb{R}$ is defined as $\Delta f = \operatorname{div}(\nabla f)$. Show the following relations for smooth functions $f, g : U \to \mathbb{R}$ and vector fields $X : U \to \mathbb{R}^n$:

   (a) $\operatorname{div}(fF) = \langle \nabla f, F \rangle + f \operatorname{div} F$,

   (b) $\Delta(fg) = f\Delta g + 2\langle \nabla f, \nabla g \rangle + g\Delta f$.

8. (Easy Ending) Which of the following maps are diffeomorphisms? Justify your answer.

   (a) $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}$, $f(x, y) = (x, x + y^3)$,

   (b) $f : (0, 1) \to (0, 1)$, $f(x) = x^2$,

   (c) $f : U_1(0) \to \mathbb{R}^2$ with $U_1(0) = \{(x, y) \mid x^2 + y^2 < 1\}$ and

$$f(x, y) = \tan(\frac{\pi}{2}(x^2 + y^2))(x, y).$$