Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 7 23.11.2011

Do **Exercise 3** as **homework for this week**. The cumulative homework over the coming weeks will be collected and marked in a few weeks time. Try to do at least one of the other exercises as well for your own benefit. All of them are very useful. Have a look at all solutions when you receive the solution sheet the following week.

In the Exercises below there appears one important concept, which will be introduced properly a bit later in the lectures for general differential forms (not only 1-forms): "A differential form ω is *closed* if $d\omega = 0$." For our purposes it suffices to use th following description of closedness: Let $U \subset \mathbb{R}^n$ be open. A 1-form $\omega = \sum_{j=1}^n f_j dx_j \in \Omega^1(U)$ is *closed* if and only if we have

$$\frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j} \qquad \text{for all } j, k \in \{1, \ldots, n\}.$$

Use this characterisation whenever a 1-form is said to be "closed" in the exercises below.

1. Let $\omega = 2xy^3\, dx + 3x^2y^2\, dy \in \Omega^1(\mathbb{R}^2)$. Show that ω is exact, i.e., there exists $f \in C^\infty(\mathbb{R}^2)$ with $\omega = df$. Calculate

$$\int_c \omega,$$

where c is the arc of the parabola $y = x^2$ from $(0,0)$ to (x,y).

2. This exercise tells us that every exact differential form is closed, but not every closed differential form is exact.

   (a) Let $U \subset \mathbb{R}^n$ be open. Show that every exact differential form $\omega \in \Omega^1(U)$ is closed.

   (b) Let $\omega_0 \in \Omega^1(\mathbb{R}^2 - 0)$ be defined as

$$\omega_0 = -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy.$$

   We showed in the lectures that ω_0 cannot be exact, since $\int_c \omega_0 \neq 0$ for certain closed curves (cf. Proposition 5.13 and Example thereafter). Check that ω_0 is closed.

3. A function $f : \mathbb{R}^3 \to \mathbb{R}$ is said to be homogeneous of degree k if $f(sx, sy, sz) = s^k f(x,y,z)$ for all $s > 0$ and $(x,y,z) \in \mathbb{R}^3$. Prove the following facts:

(a) If f is differentiable and homogeneous of degree k, then we have *Euler's relation*:

$$x\frac{\partial f}{\partial x}(x,y,z) + y\frac{\partial f}{\partial y}(x,y,z) + z\frac{\partial f}{\partial z}(x,y,z) = kf(x,y,z).$$

Hint: Differentiate $f(sx, sy, sz)$ in s and use the chain rule.

(b) If the differential form

$$\mu = u\,dx + v\,dy + w\,dz \in \Omega^1(\mathbb{R}^3)$$

is such that the coefficient functions u, v, w are homogeneous of degree k and μ is closed, then we have $\mu = df$ with

$$f = \frac{xu + yv + zw}{k+1} \in C^\infty(\mathbb{R}^2).$$

4. Let $U = \mathbb{R}^n - 0$ and $\omega \in \Omega^1(U)$ be defined by

$$\omega = \frac{1}{\|x\|_2^2} \sum_{i=1}^n x_i\,dx_i,$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Show that ω is exact. Now fix $n = 3$. Let k be an integer and $c : [0, 2k\pi] \to \mathbb{R}^3$ be the helix $c(t) = (\cos t, \sin t, t)$. Calculate $\int_c \omega$.

5. Let $c : [0,1] \to U = \mathbb{R}^2 - 0$ be a smooth closed curve, i.e., $c(1) = c(0)$ and $c(t) = r(t)(\cos\alpha(t), \sin\alpha(t))$ be its polar coordinate description with smooth functions $r : [0,1] \to (0,\infty)$ and $\alpha : [0,1] \to \mathbb{R}$. Since c is closed, the angle difference $\alpha(1) - \alpha(0)$ must be an integer multiple of 2π, i.e., $\alpha(1) - \alpha(0) = n(c)2\pi$, and the integer $n(c)$ describes how many times the curve c surrounds the origin counterclockwise. $n(c)$ is called the *winding number* of c. Let $\omega_0 \in \Omega^1(U)$ be the 1-form

$$\omega_0 = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

Show that we have

$$n(c) = \frac{1}{2\pi}\int_c \omega_0.$$

Remark: In Complex Analysis, the *winding number* of a a closed curve $c : [0,1] \to \mathbb{C} - 0$ is defined as

$$n(c) = \frac{1}{2\pi i}\int_c \frac{1}{z}\,dz.$$

This exercise presents an analogue in the context of differential forms.