Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 8 30.11.2011

Do **Exercises 2 and 3** as **homework for this week**. These homework exercises will not be marked, but you can check your solutions against the solution sheet in the following week. It is really important that you do every week the emphasized questions in order to stay up to date with the course.

1. Prove "Cramer's Rule": Let $A = (a_1, \ldots, a_n) \in GL(n, \mathbb{R})$ with linear independent columns $a_j \in \mathbb{R}^n$. Then the unique solution of $Ax = b$ with $x = (x^1, \ldots, x^n)^\top$ is given by

$$x^i = \frac{\det(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)}{\det(A)}.$$

   Hint: Note that $Ax = b$ can be rewritten as $b = \sum_j x^j a_j$, and use the fact that the determinant is multilinear and alternating.

2. Let $\omega \in \Omega^2(\mathbb{R}^3 - 0)$ be defined as

$$\omega = \frac{x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}.$$

   Show that

$$\omega_x(v, w) = \frac{\langle v \times w, x \rangle}{\|x\|^3}$$

   and calculate $d\omega$.

3. Let φ, ψ, ϕ be the following forms in $\mathbb{R}^3$:

$$\begin{aligned} \varphi &= x\,dx - y\,dy, \\ \psi &= z\,dx \wedge dy + x dy \wedge dz, \\ \phi &= z\,dz. \end{aligned}$$

   Calculate $\varphi \wedge \psi$, $\phi \wedge \varphi \wedge \psi$, $d\varphi$, $d\psi$ and $d\phi$.