**Analysis III/IV (Math 3011, Math 4201)**

**Exercise Sheet 9** 7.12.2011

Do **Exercises 2 and 3** as **homework for this week**. Since I already handed out the solution to Exercise 3 of Exercise Sheet 7, it doesn't make sense to mark it any more. Check your solution of that exercise against the Solution Sheet to Exercise Sheet 7.

So only this week's homework will be collected on Wednesday, 14 December, right after the last lecture of this term, and it will be marked over the Christmas vacation. Please do this homework, because it is important to stay up to date with the course.

1. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. We associate to $F = (f_1, f_2, f_3)$ the following differential 1- and 2-forms:

$$\omega_F = f_1 dx_1 + f_2 dx_2 + f_3 dx_3, \qquad \eta_F = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2.$$

Show the following identities:

$$\begin{aligned} df &= \omega_{\nabla f} \qquad \text{for } f \in C^\infty(\mathbb{R}^3), \\ d\omega_F &= \eta_G \qquad \text{with } G := \operatorname{curl} F : \mathbb{R}^3 \to \mathbb{R}^3, \\ d\eta_F &= \operatorname{div} F \, dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Derive form these identities and $d^2 = d \circ d = 0$ that $\operatorname{curl} \circ \nabla f = 0$ and $\operatorname{div} \circ \operatorname{curl} F = 0$.

2. Let $U \subset \mathbb{R}^n$ with $n \geq 2$ be an open set.

   (a) Show that if $\omega \in \Omega^1(U)$ and $c : [a, b] \to U$ is a smooth curve with $\|F_\omega(c(t))\| \leq M$ for all $t \in [a, b]$ (where $F_\omega : U \to \mathbb{R}^n$ is the vector field associated to $\omega$, see Lemma 5.6), then

   $$\left| \int_c \omega \right| \leq M \cdot L(c),$$

   where $L(c)$ denotes the length of the curve $c$.

   (b) Let $\omega \in \Omega^1(\mathbb{R}^n - 0)$ be a closed differential form. Assume that $\|F_\omega\|$ is bounded in some disk centered at 0. Show that $\omega$ is exact in $\mathbb{R}^n - 0$.

   **Hint:** Use the characterisation of exactness of differential 1-forms by integrals over all closed curves.

   (c) Why is the result in (b) not a contradiction to the non-exactness of the form

   $$\omega = -\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

   in Exercise 2(b) of Exercise Sheet 7?

3. This exercise is dedicated to the proof of **Poincaré's Lemma** for 1-forms on starlike open sets $U \subset \mathbb{R}^2$. Let $p \in U$ be such that, for every $x \in U$, the straight line segment connecting $p$ and $x$ lies totally in $U$. For simplicity, we assume that $p$ is the origin. The straight line segment from $p = 0$ to $x \in U$ can be parametrised by the curve $c_x : [0,1] \to U$, $c_x(t) = tx$. Assume that

$$\omega = f_1 dx_1 + f_2 dx_2 \in \Omega^1(U)$$

is closed, i.e., the functions $f_1, f_2 \in C^\infty(U)$ satisfy

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}.$$

Define $f : U \to \mathbb{R}$ by

$$f(x) = \int_{c_x} \omega.$$

The goal of this exercise is to prove $\omega = df$.

(a) Let $x = (x_1, x_2) \in U$. Show that

$$f(x) = \int_0^1 f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2 \, dt.$$

(b) Using the fact that $\omega$ is closed, prove that

$$\frac{\partial f}{\partial x_1}(x) = \int_0^1 t(f_1 \circ c_x)'(t) + f_1 \circ c_x(t) dt.$$

You are allowed to interchange the integral and partial differentiation without further justification, but carry out carefully and in detail all other steps of your calculation. Then use partial integration to prove that

$$\frac{\partial f}{\partial x_1}(x) = f_1(x),$$

and, analogously,

$$\frac{\partial f}{\partial x_2}(x) = f_2(x).$$

(c) Conclude from (b) that $\omega = df$, i.e., $\omega$ is exact.

The proof here can be easily generalised to higher dimensions and, with more effort, to $k$-forms. But it is more important that you understand the crucial ideas behind Poincaré's Lemma in a particularly easy case.