Analysis IV Extra Reading Road Map

Norbert Peyerimhoff

Durham University, 2011/12

The sources of the 4H extra reading material for Analysis IV are

- The "Notes on the Arzela-Ascoli Theorem" on Joel H. Shapiro's Page at `http://www.mth.msu.edu/~shapiro/Pubvit/LecNotes.html`
- Pages 262-285 of Kenneth Hoffman's book "Analysis in Euclidean Space"

The two sections in K. Hoffman's book *6.4. Completeness* and *6.5. Compactness* contain, besides new material, also some material which we also introduced and discussed in the lectures. The specifically new topics are *Baire Category Theorem*, *equicontinuity* and *Ascoli's Theorem*. A notion which you need to know is **countability of a set**. A set A is called *countable* if there exists a bijection between the elements of A and the natural numbers, i.e., if we can enumerate all the elements in A and obtain

$$A = \{a_1, a_2, a_3, a_4, \dots\}.$$

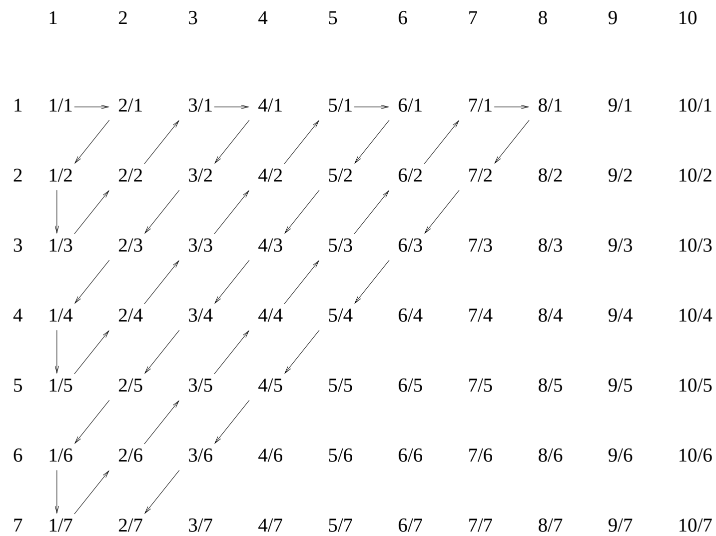
Obviously, the set of natural numbers is countable, but also the set of all integers by enumerating them as

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \dots\}$$

Even the set of all the positive rational numbers is countable, and this can be seen by producing a infinite square scheme of numerators (upper horizontal line) and denominators (left vertical column) and go through them one by one in the diagonal fashion shown on the next page, omitting a rational if it was already chosen earlier. We end up with the enumeration

$$\{1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, 6, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \frac{1}{7}, \frac{3}{5}, \frac{5}{3}, 7, 8, \frac{7}{2}, \frac{5}{4}, \frac{4}{5}, \frac{2}{7}, \dots\},$$

and one easily checks that all positive rationals are chosen during this procedure.



Once we have enumerated all positive rationals by $\{q_1, q_2, q_3, q_4, \dots\}$, we can show the countability of the set of all rationals $\mathbb{Q}$ by ordering them as

$$\mathbb{Q} = \{0, q_1, -q_1, q_2, -q_2, q_3, -q_3, \dots, \}.$$

What about the reals? Assume that **all** reals in the interval $[0, 1)$ could be enumerated (in which case $[0, 1)$ would be countable), then we would have $[0, 1) = \{x_1, x_2, x_3, \dots\}$ and we could write them in their decimal expansions one below the other:

$$\begin{aligned} x_1 &= 0.x_{11}x_{12}x_{13}x_{14}x_{15}\dots, \\ x_2 &= 0.x_{21}x_{22}x_{23}x_{24}x_{25}\dots, \\ x_3 &= 0.x_{31}x_{32}x_{33}x_{34}x_{35}\dots, \\ x_4 &= 0.x_{41}x_{42}x_{43}x_{44}x_{45}\dots, \\ &\vdots \end{aligned}$$

Georg Cantor's ingenious idea was that one could then look at a number $y \in [0, 1)$, whose decimal expansion $y = 0.y_1y_2y_3y_4y_5\dots$ is chosen to satisfy $y_1 \neq x_{11}$, $y_2 \neq x_{22}$, $y_3 \neq x_{33}$, $y_4 \neq x_{44}$, and so on. By the above assumption, y would have to coincide with some $x_N \in [0, 1)$. But the decimal expansion of y differs from that of x_N because of $y_N \neq x_{NN}$, so $y \neq x_N$, leading to a contradiction.

The diagonal arguments presented here are at the core of mathematics. They play an important role in Gödel's incompleteness proof and are simply "fundamental". In fact, The Arzela-Ascoli Theorem is also based on these diagonal arguments but, unfortunately, this is very hidden in the book by K. Hoffman. For this reason, I added the other reference "The Arzela-Ascoli Theorem" by J. H. Shapiro, because there the arguments for the proof of this Theorem are much more transparent. So focus more on this second source when trying to understand the statement and proof of Arzela-Ascoli's Theorem.