Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 11 18.1.2012

1. (a) Since ω_2 is exact, we have $\omega_2 = d\eta$. Let $\mu = \omega_1 \wedge \eta$. Assume that $\omega_1 \in \Omega^k(U)$. Then we have

$$d(\omega_1 \wedge \eta) = (d\omega_1) \wedge \eta + (-1)^k \omega_1 \wedge d\eta.$$

Since ω_1 is closed, we have $d\omega_1 = 0$, and we see that

$$d((-1)^k \omega_1 \wedge \eta) = \omega_1 \wedge \omega_2,$$

i.e., $\omega_1 \wedge \omega_2$ is exact.

(b) We have

$$\begin{aligned}
d\omega &= -y_1^2 dy_1 \wedge dy_2 \wedge dy_3, \\
\varphi^*(d\omega) &= -x_1^2 e^{2x_3}(e^{x_3}dx_1 + x_1 e^{x_3} dx_3) \wedge \\
&\quad (e^{-x_3}dx_2 - x_2 e^{-x_3} dx_3) \wedge 2x_3 dx_3 \\
&= -2x_1^2 x_3 e^{2x_3} dx_1 \wedge dx_2 \wedge dx_3, \\
\varphi^*(\omega) &= x_1^2 x_2 e^{x_3}(e^{x_3}dx_1 + x_1 e^{x_3} dx_3) \wedge 2x_3 dx_3 \\
&= 2x_1^2 x_2 x_3 e^{2x_3} dx_1 \wedge dx_3, \\
d\varphi^*(\omega) &= -2x_1^2 x_3 e^{2x_3} dx_1 \wedge dx_2 \wedge dx_3.
\end{aligned}$$

2. Homework! Will be given in a later solution sheet.

3. Note that

$$\det D\varphi(x) = \det \begin{pmatrix} \varphi_1'(x_1) & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \varphi_n'(x_n) \end{pmatrix} = \prod_i \varphi_i'(x_i).$$

Using the substitution reule in each coordinate, we obtain

$$\begin{aligned}
\int_V f &= \int_{[c_1,d_1]} \cdots \int_{[c_n,d_n]} f(y_1,\dots,y_n) dy_n \dots dy_1 \\
&= \int_{[c_1,d_1]} \cdots \int_{[a_n,b_n]} f(y_1,\dots,y_{n-1},\varphi_n(x_n)) |\varphi_n'(x_n)| dx_n dy_{n-1} \dots dy_1 \\
&\;\vdots \\
&= \int_{[a_1,b_1]} \cdots \int_{[a_n,b_n]} f(\varphi_1(x_1),\dots,\varphi_n(x_n)) \prod_i |\varphi_i'(x_i)| dx_n \dots dx_1 \\
&= \int_U f \circ \varphi |\det D\varphi|.
\end{aligned}$$

4. Homework! Will be given in a later solution sheet.

5. (a) We have $f(x, y) = (x + b, y)$ and, consequently,

$$f^*\omega = \frac{d(x+b) \wedge dy}{y^2} = \frac{dx \wedge dy}{y^2} = \omega.$$

(b) We have $g(x, y) = (ax, ay)$ and, consequently,

$$g^*\omega = \frac{d(ax) \wedge d(ay)}{(ay)^2} = \frac{a^2 dx \wedge dy}{a^2 y^2} = \omega.$$

(c) We have $h(x, y) = (x/(x^2 + y^2), -y/(x^2 + y^2))$. Note that

$$\begin{aligned} d(\frac{x}{x^2+y^2}) &= \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right) dx - \frac{2xy}{(x^2+y^2)^2} dy \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2} dx - \frac{2xy}{(x^2+y^2)^2} dy. \end{aligned}$$

Similarly, we obtain

$$d(\frac{y}{x^2+y^2}) = \frac{x^2 - y^2}{(x^2+y^2)^2} dy - \frac{2xy}{(x^2+y^2)^2} dx,$$

and

$$d(\frac{x}{x^2+y^2}) \wedge d(\frac{y}{x^2+y^2}) = -\frac{(x^2-y^2)^2 + 4x^2y^2}{(x^2+y^2)^4} dx \wedge dy = -\frac{dx \wedge dy}{(x^2+y^2)^2}.$$

This implies that

$$h^*\omega = \frac{1}{(-y/(x^2+y^2))^2} d(\frac{x}{x^2+y^2}) \wedge d(\frac{y}{x^2+y^2}) = \frac{dx \wedge dy}{y^2} = \omega.$$