Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 13 1.2.2012

1. Let $x \in \mathbb{R}^n$. Then

$$L_{(x,0),N} = \{(\lambda x, 1-\lambda) \mid \lambda \in \mathbb{R}\}$$

and $\|(\lambda x, 1-\lambda)\|_2^2 = 1$ is equivalent to $\lambda = 0$ or $\lambda = \frac{2}{1+\|x\|_2^2}$. The second equation leads to $\lambda = \frac{2}{1+\|x\|_2^2}$, which means that

$$\varphi_1(x) = \left(\frac{2x}{1+\|x\|_2^2}, \frac{\|x\|_2^2 - 1}{1+\|x\|_2^2}\right).$$

Similarly, we obtain

$$\varphi_2(x) = \left(\frac{2x}{1+\|x\|_2^2}, \frac{1-\|x\|_2^2}{\|x\|_2^2+1}\right).$$

Let $X = \frac{2x}{1+\|x\|_2^2}$ and $Z = \frac{1-\|x\|_2^2}{1+\|x\|_2^2}$. This implies that $X = (1+Z)x$ and $\varphi_2^{-1}(X,Z) = \frac{X}{1+Z}$. Consequently,

$$\varphi_2^{-1} \circ \varphi_1(x) = \varphi_2^{-1}\left(\frac{2x}{1+\|x\|_2^2}, \frac{\|x\|_2^2-1}{1+\|x\|_2^2}\right) = \frac{2x}{1+\|x\|_2^2} \cdot \frac{1+\|x\|_2^2}{2\|x\|_2^2} = \frac{x}{\|x\|_2^2}.$$

Moreover, we have

$$\frac{\partial}{\partial x_j} \frac{x_i}{\|x\|_2^2} = \frac{\delta_{ij}}{\|x\|_2^2} - \frac{2x_i x_j}{\|x\|^4}.$$

This implies that

$$D(\varphi_2^{-1} \circ \varphi_1)(x) = \frac{1}{\|x\|_2^2}\left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x\right).$$

Remark: Geometrically, the matrix $\frac{1}{\|x\|^2} x^\top x$ describes a projection on the line $\mathbb{R}v$ and $\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x \top x$ is a reflection in the hyperplane orthogonal to v. This geometric interpretation implies that $\frac{1}{\|x\|_2^2}\left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x\right)$ is an invertible matrix with inverse $\|x\|_2^2\left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x\right)$.

2. (a) We have

$$DF(x,y,z) = \left(2x\left(1 - \frac{5}{\sqrt{x^2+y^2}}\right), 2y\left(1 - \frac{5}{\sqrt{x^2+y^2}}\right), 2z\right).$$

Note that $F^{-1}(4)$ does not contain any point of the form $(0,0,z)$, since we have $F(0,0,z) = z^2 + 5^2 \geq 25$. So the preimage avoids any points which might be problematic in the formula for $DF(x,y,z)$. On the other hand, the critical points $(x,y,z) \neq 0$ are given when $z = 0$ and $x^2 + y^2 = 25$. But for those points we have $F(x,y,z) = 0 \neq 4$. This shows that 4 is a regular value of F. 4 lies in $\mathrm{im}(F)$ since we have $F(5,0,2) = 4$.

(b) We have

$$F(\varphi(\alpha,\beta)) =$$
$$(2\sin\beta)^2 + \left(\sqrt{((5+2\cos\beta)\cos\alpha)^2 + ((5+2\cos\beta)\sin\alpha)^2} - 5\right)^2 =$$
$$(2\sin\beta)^2 + (2\cos\beta)^2 = 4,$$

i.e., $\varphi(\alpha,\beta) \in M$. On the other hand, for every $(x,y,z) \in M$, we must have $3 \leq \sqrt{x^2+y^2} \leq 7$, so there exists $\alpha \in [0,2\pi)$ and $3 \leq \rho \leq 7$ with

$$(x,y) = \rho(\cos\alpha, \sin\alpha).$$

On the other hand, we must have $z^2 + (\rho - 5)^2 = 4$, i.e., there is a $\beta \in [0,2\pi)$ such that $(\rho - 5, z) = 2(\cos\beta, \sin\beta)$. Both results together imply that $z = 2\sin\beta$ and $\rho = 5 + 2\cos\beta$ and $(x,y) = (5+2\cos\beta)(\cos\alpha, \sin\alpha)$, i.e.,

$$M = \{((5+2\cos\beta)\cos\alpha, (5+2\cos\beta)\sin\alpha, 2\sin\beta) \mid \alpha,\beta \in [0,2\pi)\}.$$

This implies that the points of M, not covered by $\varphi(U)$, are the (closed) curves

$$c_1(t) = (5+2\cos\beta, 0, 2\sin\beta), \quad t \in [0,2\pi]$$

and

$$c_2(t) = (7\cos\alpha, 7\sin\alpha, 0), \quad t \in [0,2\pi].$$

Therefore, φ is an almost global coordinate patch of M.

(c) We have

$$\begin{aligned}
\varphi^* dx &= -(5+2\cos\beta)\sin\alpha\, d\alpha - 2\sin\beta\cos\alpha\, d\beta, \\
\varphi^* dy &= (5+2\cos\beta)\cos\alpha\, d\alpha - 2\sin\beta\sin\alpha\, d\beta, \\
\varphi^* dz &= 2\cos\beta\, d\beta, \\
\varphi^*(dy \wedge dz) &= 2(5+2\cos\beta)\cos\alpha\cos\beta\, d\alpha \wedge d\beta, \\
\varphi^*(dx \wedge dz) &= -2(5+2\cos\beta)\sin\alpha\cos\beta\, d\alpha \wedge d\beta, \\
\varphi^*(dx \wedge dy) &= 2(5+2\cos\beta)\sin\beta\, d\alpha \wedge d\beta, \\
\varphi^*\omega &= 2(5+2\cos\beta)(5\cos\beta + 2)\, d\alpha \wedge d\beta.
\end{aligned}$$

3. We can cover M with one global coordinate patch, namely $\varphi : U \to \mathbb{R}^{k+1}$, $\varphi(x) = (x, f(x))$. φ is obviously continuous and we have $M = \mathrm{im}(\varphi)$. Moreover, we have $\varphi^{-1}(x, y) = x$, which is again obviously continuous. Finally, the Jacobi matrix of φ is given by

$$D\varphi(x) = \begin{pmatrix} \mathrm{Id}_k \\ \frac{\partial f}{\partial x_1}(x) \cdots \frac{\partial f}{\partial x_k}(x) \end{pmatrix},$$

which has obviously rank k. This shows that φ has all properties of a global coordinate patch and M is a smooth manifold.

4. (a) Let f be a homogeneous polynomial of degree $m \geq 1$ and $y \neq 0$. Let $x \in f^{-1}(y)$. Then we obtain from Euler's relation:

$$\langle \mathrm{grad} f(x), x \rangle = m f(x) = m y \neq 0.$$

This implies that $\mathrm{grad} f(x) \neq 0$, so $Df(x) : \mathbb{R}^k \to \mathbb{R}$ is surjective for all $x \in f^{-1}(y)$. Therefore, $y \neq 0$ is a regular value.

(b) The group $SL(n, \mathbb{R}) \subset M(n, \mathbb{R}) = \mathbb{R}^{n^2}$ ist equal to $f^{-1}(1)$, where $f(A) = \det A$. Now, f is a homogeneous polynomial of degree n in $\mathbb{R}^{n^2}$, so 1 is a regular value of f, by (a). Theorem 9.5 implies that $SL(n, \mathbb{R}) = f^{-1}(1)$ is a differentiable manifold of dimension $n^2 - 1$.

Finally, we provide the solutions for the homeworks:

2. (From Exercise Sheet 11) We have

$$\begin{aligned} f^*(y_3 dy_1 \wedge dy_2 \wedge dy_3) &= x_3 d(x_1 \cos x_2) \wedge d(x_1 \sin x_2) \wedge dx_3 \\ &= x_3(\cos x_2 dx_1 - x_1 \sin x_2 dx_2) \wedge (\sin x_2 dx_1 + x_1 \cos x_2 dx_2) \wedge dx_3 \\ &= x_3(x_1 \cos^2 x_2 + x_1 \sin^2 x_2) dx_1 \wedge dx_2 \wedge dx_3 \\ &= x_1 x_3 dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

This implies that

$$\begin{aligned} \int_{(1,2)\times(0,2\pi)\times(0,1)} \omega &= \int_1^2 \int_0^{2\pi} \int_0^1 x_1 x_3 dx_3 dx_2 dx_1 \\ &= 2\pi \int_1^2 \left[\frac{1}{2} x_1 x_3^2\right]_{x_3=0}^{x_3=1} dx_1 \\ &= \pi \int_1^2 x_1 dx_1 = \frac{3}{2}\pi. \end{aligned}$$

4. (From Exercise Sheet 11) For each i, choose a countable set of rectangles $Q_1^i, Q_2^i, \ldots$ such that

$$A_i \subset \bigcup_j Q_j^i$$

and

$$\sum_j v(Q_j^i) < \frac{\epsilon}{2^i}.$$

Then we have

$$\bigcup_i A_i \subset \bigcup_{i,j} Q_j^i$$

and

$$\sum_{i,j} v(Q_j^i) < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

Moreover, the set of rectangles Q_j^i is countable, since we can enumerate them by Q_1^1, Q_2^1, Q_1^2, Q_3^1, Q_2^2, Q_1^3... I.e., we choose first all rectangles Q_j^i where $i+j$ adds up to 1, then the ones where $i+j$ adds up to 2, then the ones where $i+j$ adds up to 3,... In this way, we capture each one of the Q_j^i's in our enumeration. This shows that $\bigcup_i A_i$ is also a set of measure zero.