Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 4 31.10.2011

1. We have to check the norm axioms. Firstly, $\|x\| = 0$ is equivalent to $\|x\|_1 = 0$ and $\|x\|_2 = 0$, which is true iff $x = 0$. Secondly,

$$\begin{aligned}\|\lambda x\| &= \alpha|\lambda|\|x\|_1 + \beta|\lambda|\|x\|_2 \\ &= |\lambda|\left(\alpha\|x\|_1 + \beta\|x\|_2\right) = |\lambda|\|x\|.\end{aligned}$$

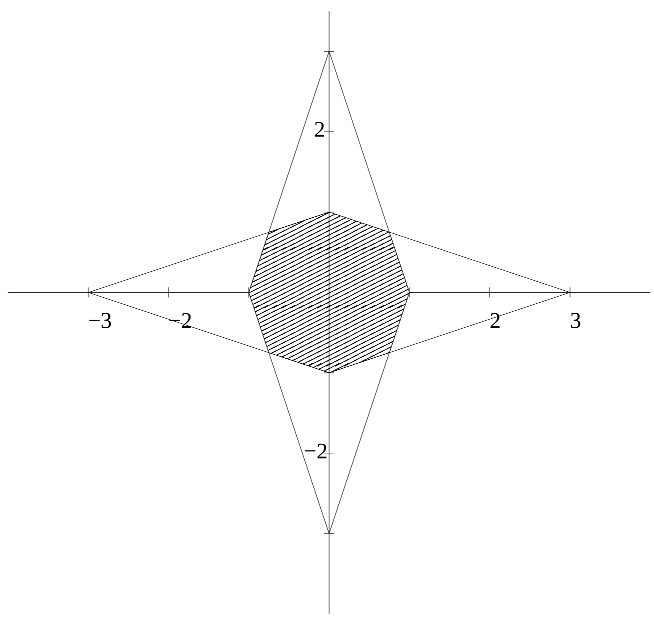
Finally,

$$\begin{aligned}\|x+y\| &\leq \alpha(\|x\|_1 + \|y\|_1) + \beta(\|x\|_2 + \|y\|_2) \\ &= (\alpha\|x\|_1 + \beta\|x\|_2) + (\alpha\|y\|_1 + \beta\|y\|_2) = \|x\| + \|y\|.\end{aligned}$$

For the concrete norm we have the equivalences

$$\begin{aligned}\|x\| \leq 1 \quad &\Leftrightarrow \quad \frac{1}{3}(|x_1| + |x_2|) + \frac{2}{3}|x_1| \leq 1 \,\text{and} \\ &\qquad\; \frac{1}{3}(|x_1| + |x_2|) + \frac{2}{3}|x_2| \leq 1 \\ &\Leftrightarrow \quad 3|x_1| + |x_2| \leq 3 \,\text{and}\, |x_1| + 3|x_2| \leq 3,\end{aligned}$$

so the shape of the unit ball looks as follows (shaded area):



2. Let $A_n := \sum_{k=1}^{n} a_k$. If A_n is convergent, then it is a Cauchy. This means that for every $\epsilon > 0$ there exists n_0 such that for all $n \geq m \geq n_0$:

$$\nu_p(A_n - A_m) < \epsilon.$$

This shows that, in particular, $\nu_p(a_{m+1}) = \nu_p(A_{m+1} - A_m) < \epsilon$, for all $m \geq n_0$, which means $a_k \to 0$. Conversely, let us assume that $a_k \to 0$. We need to show that A_n is Cauchy. We conclude from the strong triangle inequality that

$$\nu_v(A_n - A_m) = \nu_v(\sum_{k=m+1}^{n} a_k) \leq \max\{\nu_p(a_{m+1}), \nu_p(a_{m+1}), \dots \nu_p(a_n)\}.$$

For every $\epsilon > 0$, there exists n_0 such that $\nu_p(a_n) < \epsilon$ for all $n \geq n_0$. This implies that, for all $n > m \geq n_0$:

$$\nu_v(A_n - A_m) < \epsilon,$$

i.e., A_n is Cauchy.

3. It is easy to see that $\mathcal{B}(V, W)$ is a vector space and that the operator norm is actually a norm on this vector space. We focus on proving that if $T_n \in \mathcal{B}(V, W)$ is a Cauchy sequence, then there exists an operator $T \in \mathcal{B}(V, W)$ such that $T_n \to T$, i.e., $\|T_n - T\| \to 0$. We first have to define the limit operator $T : V \to W$. Let $v \in V$. Then $w_n := T_n v \in W$ is a Cauchy sequence because of

$$\|w_n - w_m\|_W = \|T_n v - T_m v\|_W \leq \|T_n - T_m\| \cdot \|v\|_V$$

and the fact that T_n is a Cauchy sequence. Since $(W, \|\cdot\|_W)$ is a Banach space, $w_n \in W$ must be convergent and we define

$$Tv = \lim_{n\to\infty} w_n = \lim_{n\to\infty} T_n v.$$

This defined the operator T pointwise. Let us first check that T is linear:

$$\begin{aligned} T(v_1 + v_2) &= \lim T_n(v_1 + v_2) = \lim T_n v_1 + T_n v_2 \\ &= \lim T_n v_1 + \lim T_n v_2 = Tv_1 + Tv_2, \\ T(\lambda v) &= \lim T_n(\lambda v) = \lim \lambda T_n(v) = \lambda \lim T_n(v) = \lambda T v. \end{aligned}$$

Next, we need to show that T is bounded. Since T_n is a Cauchy sequence, T_n is bounded (see Exercise 3 on Sheet 1), i.e., there exists $C > 0$ such that $\|T_n\| \leq C$ for all n. Let $v \in V$ with $\|v\|_V \leq 1$. Since $T_n v \to Tv$ there exists n_0 such that $\|Tv - T_{n_0} v\|_W < 1$. This implies that

$$\|Tv\|_W \leq \|Tv - T_{n_0} v\|_W + \|T_{n_0} v\|_W < 1 + \|T_{n_0}\| \cdot \|v\|_V \leq 1 + C,$$

i.e., $T \in \mathcal{B}(V, W)$. It only remains to show that $T_n \to T$. Let $\epsilon > 0$ be given. Since T_n is a Cauchy sequence, there exists a n_0 such that $\|T_n - T_m\| < \epsilon/2$ for all $n, m \geq n_0$. Let $v \in V$ with $\|v\|_V \leq 1$. Since $T_n v \to Tv$, there exists $n_0(v)$ such that $\|T_n v - Tv\|_W < \epsilon/2$ for all $n \geq n_0(v)$. We can assume, without loss of generality, that $n_0(v) \geq n_0$. Then we have for all $n \geq n_0$:

$$\begin{aligned} \|Tv - T_n v\|_W &\leq \|Tv - T_{n_0(v)} v\|_W + \|T_{n_0(v)} v - T_n v\|_W \\ &< \epsilon/2 + \|T_{n_0(v)} - T_n\| \cdot \|v\|_V < \epsilon/2 + \epsilon/2 \cdot 1 = \epsilon. \end{aligned}$$

This shows that $\|T - T_n\| < \epsilon$ for all $n \geq n_0$, i.e., $T_n \to T$.

4. (i) Let $f(x) = \frac{1}{e^{cb}} e^{Cx}$. Then $\|f\|_\infty = 1$ and $f'(x) = Cf(x)$. Therefore, we have

$$\|Df\|_\infty = C.$$

Since $C > 0$ can be arbitrarily, D is unbounded.

(ii) The fastest way to show boundedness of the restricted operator D is to identify $P_k[a,b]$ with the vector space $\mathbb{R}^{k+1}$ via

$$a_k x^k + \cdots + a_1 x + a_0 \mapsto (a_k, \ldots, a_1, a_0).$$

By this identification, D translates into the linear operator

$$D(a_k, \ldots, a_1, a_0) = (0, ka_k, \ldots, 2a_2, a_1),$$

which can be written as a matrix, if required. Since all norms in $\mathbb{R}^{k+1}$ are equivalent and every matrix is a bounded operator with respect to any norm, we conclude that D is bounded on $P_k[a,b]$.

(iii) Let $f \in C^1[a,b]$ be a non-vanishing constant function. Then $\|f\|_* = 0$, but $f \neq 0$, a contradiction to the norm axioms.

(iv) This is not a norm on $C[0,1]$ for similar reasons as in (iii). Choose a non-vanishing continuous function f on $[0,1]$ which vanishes at the $k+1$ points $\frac{j}{k}$ for $j = 0, 1, \ldots, k$. Then $\|f\|_\Delta = 0$ but $f \neq 0$. Such an example does not exist for polynomials of degree $\leq k$. If $p \in P_k[0,1]$ and $\|p\|_\Delta = 0$, we conclude that p has $k+1$ distinct zeroes on the real line. Since p is of degree k, it cannot have more than k zeros, unless it is identically zero. This shows $\|p\|_\Delta = 0 \Leftrightarrow p = 0$ in $P_k[0,1]$. The other norm axioms are easily checked.

(v) The norm axioms are easily checks, only $\|f\|_{c^1} = 0 \Leftrightarrow f = 0$ needs to be considered. But this follows immediately from $\|f\|_\infty = 0 \Leftrightarrow f = 0$. The boundedness of D is shown as follows: Let $f \in C^1[a,b]$ with $\|f\|_{C^1} \leq 1$. Then

$$\|D(f)\|_\infty = \|f'\|_\infty \leq \|f\|_{C^1} \leq 1,$$

i.e., D is bounded.

(iv) The norm axioms are easily checked, only $\|f\|_\diamond = 0 \Leftrightarrow f = 0$ needs consideration. If $\|f\|_\diamond = 0$ we have $f' = 0$. Since $[a,b]$ is a connected set, f must be a constant function. Since $f(a) = 0$, f must vanish everywhere. This shows $f = 0$. The converse direction is trivial.

5. (i) Look at $g(x) = f(x) - x$. Then $g(a) \geq 0$ and $g(b) \leq 0$, so there must be $x \in [a,b]$ with $g(x) = 0$. This implies $f(x) = x$.

(ii) Since $|f'(x)| < 1$ for all $x \in [a,b]$ and $\|f'(x)\|$ is continuous on $[a,b]$, it attains a maximum M on $[a,b]$ with $M < 1$. Using the Mean Value Theorem, we obtain

$$\|f(x) - f(y)\| \leq \|f'(\xi)\| \cdot |x - y| \leq M \cdot |x - y|,$$

for some ξ between x and y. This means that $f : [a, b] \to [a, b]$ is a contraction on the complete metric space

$$(M, d) = ([a, b], d(x, y) = |x - y|).$$

The statement of the exercise is then just an application of the Contraction Mapping Principle.

(iii) Choose $f(x) = a + b - x$. Then $f'(x) = -1$. Choose, e.g., $x_0 = a$, then we have $x_n = b$ for all odd n and $x_n = a$ for all even n.