Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 5 7.11.2011

1. (x_n) is Cauchy: Note that $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} < \infty$. For every $\epsilon > 0$ there is n_0 such that $\sum_{j=n_0}^{\infty} \frac{1}{j^2} < \epsilon$ and therefore, for $n, m \geq n_0$, $m \geq n$:

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \leq sum_{j=n}^{m-1} \frac{1}{j^2} \leq sum_{j=n_0}^{\infty} \frac{1}{j^2} < \epsilon.$$

Since (M, d) is complete, (x_n) is convergent.

2. Choose the ray $(x, y) = t(\cos\theta, \sin\theta)$ with $t > 0$. Then

$$f(x, y) = \frac{t^2 \cos\theta \sin\theta}{t^2} = \frac{1}{2}\sin(2\theta).$$

Therefore, we can find rays on which f assumes any fixed value between $-1/2$ and $1/2$. If f were continuous in the origin, the limit of f along all rays at $(x, y) = 0$ would have to be 0. Contradiction!

The first partial derivatives are obviously well defined at all points $(x, y) \neq 0$. Let $(x, y) = 0$. Then

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Similarly, $\frac{\partial f}{\partial y}(0, 0) = 0$. If both partial derivatives would be continuous at $(x, y) = 0$, then f would be totally differentiable there and therefore, also continuous. But we showed that f is not continuous at the origin.

3. Let $f, g \in C(I_\epsilon)$. We have

$$\begin{aligned} \|Tf(t) - Tg(t)\| &\leq \int_{t_0}^{t} |F(f(s), s) - F(g(s), s)| \, ds \\ &\leq L \int_{t_0}^{t} |f(s) - g(s)| \, ds \leq L \cdot \epsilon \cdot \|f - g\|_\infty. \end{aligned}$$

This implies that

$$d_\infty(Tf, Tg) = \|Tf - Tg\|_\infty \leq L\epsilon \|f - g\|_\infty = L\epsilon d_\infty(f, g).$$

If we choose $\epsilon < 1/L$, we see that T is a contraction.

4. We prove the equivalence by $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$:

$(i) \Rightarrow (ii)$: Let T be bounded, i.e., $\|Tv\|_W \leq C\|v\|_V$ for all $v \in V$. Let $v_n \in V$ with $v_n \to 0$. This means that $\|v_n\|_V \to 0$. We conclude that

$$\|T(v_n) - T(0)\|_W = \|T(v_n)\|_W \leq C\|v_n\|_V \to 0.$$

Continuity at $v = 0$ follows now from Proposition 1.24.

(ii) $\Rightarrow$ *(iii)*: Let T be continuous at 0. Let $v_n \to v \in V$. Then $v_n - v \to 0$. We know from the continuity at 0 that $T(v_n - v) \to 0$. But this means that $T(v_n) - T(v) \to 0$, or $T(v_n) \to T(v)$. This is continuity at v, again by Proposition 1.24.

(iii) $\Rightarrow$ *(i)*: We prove that the negation of *(i)* contradicts to the continuity of T. Let T be unbounded. Then there exists a sequence $v_n \in V$ with $\|v_n\|_V \le 1$ such that $\|T(v_n)\| \ge n$. Let $x_n = \frac{1}{n} v_n$. Then we have $\|x_n\| \le 1/n \to 0$, i.e., $x_n \to 0$. If T were continuous at 0, we would have $T(x_n) \to T(0) = 0$, and therefore $\|T(x_n)\|_W \to 0$. But

$$\|T(x_n)\|_W = \frac{1}{n} \|T(v_n)\|_W \ge 1.$$

This is a contradiction.

5. (i) The inequality holds trivially if $\mathbf{x} = 0$ or $\mathbf{y} = 0$. So we assume that both sequences are not zero. Note that $\sum_{n=1}^{\infty} \xi_n = 1 = \sum_{n=1}^{\infty} \eta_n$. Applying (1) to ξ_n and η_n yields

$$\frac{1}{\|\mathbf{x}\|_p \cdot \|\mathbf{y}\|_q} \sum_{n=1}^{\infty} |x_n y_n| = \sum_n \xi_n^{1/p} \eta_n^{1/q} \le \sum_n \frac{\xi_n}{p} + \frac{\eta_n}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

(ii) We know that $(x_n + y_n) \in l_p(\mathbb{C})$, i.e., $\sum_n |x_n + y_n|^p < \infty$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we conclude that $q = \frac{p}{p-1}$ and

$$\sum_n |z_n|^q = \sum_n (|x_n + y_n|^{p-1})^{\frac{p}{p-1}} = \sum_n |x_n + y_n|^p < \infty,$$

i.e., $\mathbf{z} = (z_n) \in l_q(\mathbf{C})$.

(iii) We have

$$\|\mathbf{x}+\mathbf{y}\|_p^p = \sum_n |x_n + y_n| \cdot |x_n + y_n|^{p-1} \le \sum_n |x_n| \cdot |z_n| + \sum_n |y_n| \cdot |z_n|.$$

Applying Hölder on the right hand side, we obtain

$$\|\mathbf{x}+\mathbf{y}\|_p^p \le (\|\mathbf{x}\|_p + \|\mathbf{y}\|_q) \cdot \left(\sum_n |x_n + y_n|^{(p-1)q} \right)^{1/q}.$$

Note that $(p-1)q = p$ and $1/q = (p-1)/p$. Therefore,

$$\|\mathbf{x}+\mathbf{y}\|_p^p \le (\|\mathbf{x}\|_p + \|\mathbf{y}\|_p) \cdot \|\mathbf{x}+\mathbf{y}\|_p^{p-1}.$$

Note that if $\|\mathbf{x}+\mathbf{y}\|_p = 0$, Minkowski's Inequality is trivial. Therefore, we assume that $\|\mathbf{x}+\mathbf{y}\|_p \neq 0$, and we can divide the previous inequality by $\|\mathbf{x}+\mathbf{y}\|_p^{p-1}$ and obtain

$$\|\mathbf{x}+\mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

Finally, we provide the solutions for the homeworks:

2. (From Exercise Sheet 1) (a) We only need to check for $x_0 = 1 - 1/n$ that

$$\frac{n}{2}x_0 - \frac{n-1}{2} = 0,$$

and for $x_1 = 1 + 1/n$ that

$$\frac{n}{2}x_1 - \frac{n-1}{2} = 1.$$

This is obviously true.

(b) Note that $0 \leq f_n(x) \leq 1$ and, consequently $|f_n(x) - f_m(x)|^2 \leq 1$. Moreover, for $n, m \geq n_0$, the two functions f_n, f_m can only differ in the interval $(1 - 1/n_0, 1 + 1/n_0)$. We obtain

$$\begin{aligned} d(f_n, f_m)^2 &= \langle f_n - f_m, f_n - f_m \rangle \\ &= \int_{1-1/n_0}^{1+1/n_0} |f_n(x) - f_m(x)|^2 dx \leq \frac{2}{n_0} \to 0 \qquad \text{as } n_0 \to \infty. \end{aligned}$$

This shows that f_n is a Cauchy sequence.

It is convincing that the sequence f_n, if convergent, would have to have the limit

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x \in (1,2] \end{cases}$$

and the value of f at $x = 1$ could be anything. But this function would not be continuous and, therefore, V cannot be complete.

2. (From Exercise Sheet 3) Assume f is not uniformly continuous. Then there exists an $\epsilon > 0$ such that for every δ there exists a pair $x = x(\delta)$ and $y = y(\delta)$ such that $d(x, y) < \delta$ and $d(f(x), f(y)) \geq \epsilon$. Choosing $\delta = 1/n$, we obtain a sequence x_n, y_n with $d(x_n, y_n) < 1/n$ and $d(f(x_n), f(y_n)) \geq \epsilon$. Since M is compact, we can choose a convergent subsequence $x_{n_j} \to x_0 \in M$. Since $d(x_n, y_n) < 1/n$, we also have $y_{n_j} \to x_0 \in M$. Since f is continuous in x_0, there exists a δ_0 such that $d(f(x_0), f(y)) < \epsilon/2$ for all $y \in M$ with $d(x_0, y) < \delta_0$. Since both x_{n_j} and y_{n_j} converge to x_0, there must exist a j_0 such that $d(x_0, x_{n_j}) < \delta_0$ and $d(x_0, y_{n_j}) < \delta_0$ for all $j \geq j_0$. This implies, for $j \geq j_0$,

$$\epsilon \leq d(f(x_{n_j}), f(y_{n_j})) \leq d(f(x_{n_j}), f(x_0)) + d(f(x_0), f(y_{n_j})) < \epsilon/2 + \epsilon/2 = \epsilon,$$

a contradiction.