Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 6 16.11.2011

1. We conclude from $f(x+h)-f(x) = Ah = Df(x)h+R(h)$ that $Df(x) = A$ and $R = 0$. Moreover,

$$g(x+h) - g(x) = \langle h, Ax\rangle + \langle x, Ah\rangle + \langle h, Ah\rangle = \langle (A + A^\top)x, h\rangle + \langle h, Ah\rangle$$

implies that $g(x+h)-g(x) = x^\top(A+A^\top)h+R(h)$ with $R(h) = \langle h, AH\rangle$. Since

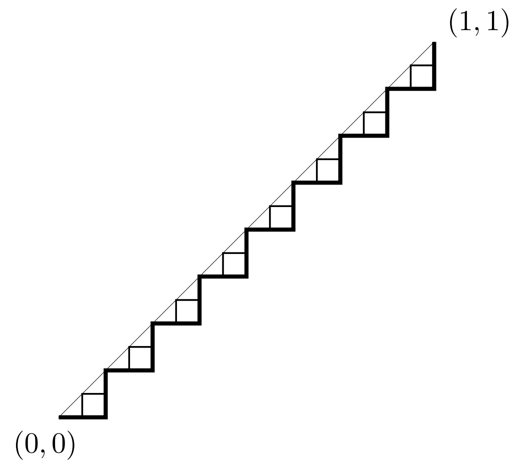
$$0 \le \lim_{h\to 0} \frac{\|R(h)\|_2}{\|h\|_2} \le \lim_{h\to 0} \frac{\|A\| \cdot \|h\|_2^2}{\|h\|_2} = 0,$$

we see that the error term $R(h)$ behaves the right way and we have $Df(x) = x^\top(A + A^\top)$.

2. The statement is false. Choose $c : [0,1] \to \mathbb{R}^2$, $c(t) = (t,t)$, and $c_n : [0,1] \to \mathbb{R}^2$ to be an approximation, which is piecewise defined and looking like a staircase with 2^n steps. The functions c, c_3 and c_4 are illustrated below. Obviously, we have

$$\max_{t\in[a,b]} \|c(t) - c_n(t)\|_2 \to 0 \qquad \text{as } n \to \infty,$$

but $L(c) = \sqrt{2}$ and $L(c_n) = 2$ for all n.

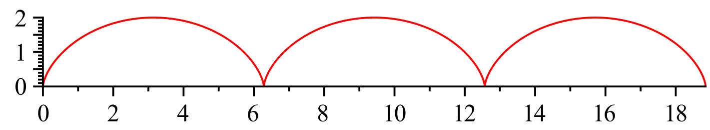


3. (a) We apply the chain rule to $f(\gamma(t)) = c$ and obtain

$$0 = \frac{d}{dt} f(\gamma(t)) = Df(\gamma(t))(\gamma'(t)) = \langle \nabla f(\gamma(t)), \gamma'(t)\rangle.$$

(b) Let $S_r = \{x \in \mathbb{R}^2 \mid \|x\| = r\}$ and $\gamma : [a,b] \to S_r$ be a curve in the sphere S_r. It suffices to show that $g \circ \gamma$ is constant, which is equivalent to

$$\frac{d}{dt} g(\gamma(t)) = 0.$$



Using, again, the chain rule, we obtain

$$\frac{d}{dt}g(\gamma(t)) = \langle \nabla g(\gamma(t)), \gamma'(t)\rangle = h(\gamma(t))\langle \gamma(t), \gamma'(t)\rangle.$$

Since $\|\gamma(t)\|^2 = \langle \gamma(t), \gamma(t)\rangle = r^2$, we conclude that

$$0 = \frac{d}{dt}\langle \gamma(t), \gamma(t)\rangle = 2\langle \gamma(t), \gamma'(t)\rangle.$$

Put together, this implies $\frac{d}{dt}g(\gamma(t)) = 0$, which we wanted to show.

4. Three full turns of the cycloid are illustrated above. Since $c'(t) = (r - r\cos(t), r\sin(t))$ we have

$$\|c'(t)\|^2 = 2r^2(1-\cos(t)) = 4r^2\sin^2(t/2),$$

and the required length is given by

$$2r\int_0^{2\pi}\sin(t/2)dt = 4r\int_0^{\pi}\sin(t)dt = 8r.$$

5. We have

$$\frac{\partial F}{\partial y}(x,y) = 4y(1+x^2+y^2),$$

i.e., $\frac{\partial F}{\partial y}$ vanishes precisely at $y = 0$. Looking at the level sets, they have vertical tangents at all points $(x,y) = (x,0) \neq (0,0)$ and, therefore, the y-coordinate cannot be decribed, locally near these points, as function of the x-coordinate. At $(x,y) = (0,0)$, the lemniscate has a cross-over and, again, it is not possible to describe the y-coordinate of the lemniskate, as a function of the x-coordinate near the origin (every x-value near 0 would correspond to two y-values).

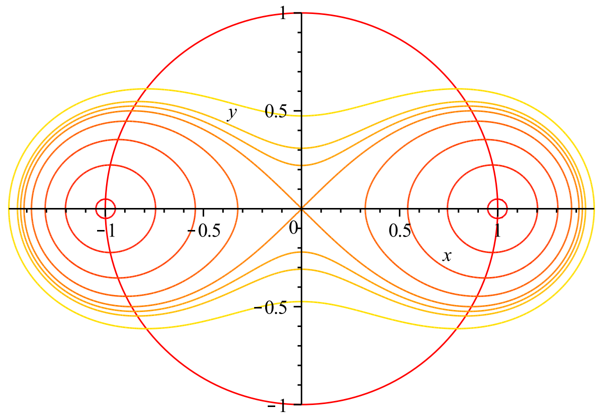
Assuming y as a function of x in a level set (which means we exclude $y = 0$), we obtain by differentiation:

$$0 = 2(x^2+y^2)(2x+2yy') - 4x + 4yy',$$

i.e.,

$$y' = \frac{x(1-x^2-y^2)}{y(1+x^2+y^2)}.$$

This shows that y' vanishes at $x = 0$ and at $x^2+y^2 = 1$, which describes a unit circle. The picture, again, illustrates that the y-coordinate of the level sets assumes its maximal and minimal value at its intersection with the circle $x^2+y^2 = 1$.



6. We have $Df(x_1, x_2) = \begin{pmatrix} 2x_1 & 0 \\ 1 & 3x_2^2 \end{pmatrix}$. If f were locally invertible at $x = (0,0)$, $Df(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ would have to be an invertible matrix, which it is not. Since $Df(1,1) = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ is invertible, f is locally invertible at $x = (1,1)$, by the Inverse Function Theorem. Moreover,

$$Df^{-1}(1,2) = Df^{-1}(f(x)) = (Df(x))^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}.$$

7. (a) We have

$$\begin{aligned} \operatorname{div}(fF)(x) = \sum_{i=1}^{n} \frac{\partial(fF_i)}{\partial x_i}(x) &= \sum_i \frac{\partial f}{\partial x_i} F_i(x) + f(x)\frac{\partial F_i}{\partial x_i}(x) \\ &= \langle \nabla f(x), F(x)\rangle + f(x)\operatorname{div} F(x). \end{aligned}$$

(b) Using the product rule, we obtain

$$\nabla(fg)(x) = \left(\frac{\partial(fg)}{\partial x_1}(x), \ldots, \frac{\partial(fg)}{\partial x_n}(x)\right) = f(x)\nabla g(x) + g(x)\nabla f(x).$$

This implies with (a)

$$\Delta(fg) = \operatorname{div}\nabla(fg) = \operatorname{div}(f\nabla g) + \operatorname{div}(g\nabla f) = f\Delta g + 2\langle \nabla f, \nabla g\rangle + g\Delta f.$$

8. (a) f is not a diffeomorphism even though it is bijective: If $(x, x+y^3) = (x_1, x_1 + y_1^3)$, then $x = x_1$ and $y^3 = y_1^3$, and the injectivity follows

from the injectivity of the function $y \mapsto y^3$ on the reals. For the surjectivity, we have to solve $(x, x+y^3) = (u, v)$, which yields $x = u$ and $y^3 = v - u$. The latter has a solution because $y \mapsto y^3$ on the reals is surjective. If $(u, v) \neq (0, 0)$, then $u \neq 0$ (in which case $x \neq 0$ and $(x, y) \neq (0, 0)$) or $u = 0$ and $v \neq 0$ (in which case $y^3 = v \neq 0$ and $(x, y) \neq (0, 0)$). But if f were a diffeomorphism, its Jacobi matrix $Df(x, y)$ would have to be invertible for all $(x, y) \neq (0, 0)$ (since $(Df(x, y))^{-1} = Df^{-1}(f(x, y))$). we have

$$Df(x, y) = \begin{pmatrix} 1 & 0 \\ 1 & 3y^2 \end{pmatrix},$$

which is obviously not invertible whenever $y = 0$.

(b) f is obviously bijective and we have $f'(x) = 2x > 0$ for all $x \in (0, 1)$. The same argument as above shows that f is a diffeomorphism.

(c) f is not a diffeomorphism, since we have $\det Df(0, 0) = 0$: Let $t = \tan(\frac{\pi}{2}(x^2 + y^2))$, for simplicity. Then

$$\begin{aligned}
\frac{\partial f_1}{\partial x}(x, y) &= \pi(1 + t^2)x^2 + t, \\
\frac{\partial f_1}{\partial y}(x, y) &= \pi(1 + t^2)xy, \\
\frac{\partial f_2}{\partial x}(x, y) &= \pi(1 + t^2)xy, \\
\frac{\partial f_2}{\partial y}(x, y) &= \pi(1 + t^2)y^2 + t.
\end{aligned}$$

This implies that $\det Df(x, y) = \pi t(1 + t^2)(x^2 + y^2) + t^2$. Note that $t = 0$ for $(x, y) = (0, 0)$, i.e., $\det Df(0, 0) = 0$.