Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 8 30.11.2011

1. Since the columns $a_1, \dots, a_n$ of A are linear independent, they form a basis of $\mathbb{R}^n$. Thus a given $b \in \mathbb{R}^n$ can be expressed as a linear combination of these columns, i.e., there exist coefficients $x^1, \dots, x^n \in \mathbb{R}$ such that

$$b = \sum_j x^j a_j.$$

Using this identity, we calculate $\det A_i := \det(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$: By multilinearity we have

$$\begin{aligned} \det A_i &= \det(a_1, \dots, a_{i-1}, \sum_j x^j a_j, a_{i+1}, \dots, a_n) \\ &= \sum_j x^j \det(a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n). \end{aligned}$$

Since the determinant is alternating, we have

$$\det(a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n) = 0$$

for all $j \neq i$, and

$$\det(a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n) = \det A$$

for $j = i$. Therefore, we obtain

$$\det A_i = x^i \det A.$$

Note that $\det A \neq 0$, so $x^i = \det A_i / \det A$.

This shows that if $b = \sum_j x^j a_j = Ax$, then the coefficients x^i can be gained back via the formula $x^i = \det A_i / \det A$, proving Cramer's Rule.

2. The first part of the exercise boils down to

$$\begin{aligned} v \times w &= (dx_2 \wedge dx_3(v, w), dx_3 \wedge dx_1(v, w), dx_1 \wedge dx_2(v, w))^\top \\ &= (v_2 w_3 - w_2 v_3, v_3 w_1 - w_3 v_1, v1w2 - w1v2)^\top, \end{aligned}$$

which is obviously correct. For the second part of the exercise, write

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$

with

$$f_i = \frac{x_i}{(x_1^2 + x_2^2 + x_3)^{3/2}}.$$

This implies

$$d\omega = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{x_3} \right) dx_1 \wedge dx_2 \wedge dx_3.$$

Since

$$\frac{\partial f_i}{\partial x_i} = \frac{1}{\|x\|^3} - \frac{3x_i^2}{\|x\|^5},$$

we conclude that

$$\sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = \frac{3}{\|x\|^3} - \frac{3\sum_i x_i^2}{\|x\|^5} = 0,$$

i.e., $d\omega = 0$.

3. We have

$$\begin{aligned} \varphi \wedge \psi &= x^2 dx \wedge dy \wedge dz, \\ \phi \wedge \varphi \wedge \psi &= 0, \\ d\varphi &= 0, \\ d\psi &= 2dx \wedge dy \wedge dz, \\ d\phi &= 0. \end{aligned}$$