Differential Geometry III Problems
Michaelmas 2012

I. Plane curves

1. Sketch the trace of the smooth curve given by $\underline{\alpha}(u) = (u^3, u^2)$, and mark the singular points.

2. An *epicycloid* $\underline{\alpha}$ is obtained as the locus of a point on the circumference of a circle of radius r which rolls without slipping on a circle of the same radius. Show (but only if you are feeling ambitious!) that the epicycloid may be parametrized by

$$\underline{\alpha}(u) = (2r \sin u - r \sin 2u,\ 2r \cos u - r \cos 2u), \qquad u \in \mathbf{R}.$$

   Find the length of $\underline{\alpha}$ between the singular points at $u = 0$ and $u = 2\pi$.
   [Use $\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$, and $\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$]

3. The *catenary* is the plane curve $\underline{\alpha}(u)$ given by $\underline{\alpha}(u) = (u, \cosh u)$. It is the curve assumed by a uniform chain hanging under the action of gravity. Sketch the curve. Find its curvature.

4. The *tractrix* is the plane curve $\underline{\alpha}(u)$ given by

$$\underline{\alpha}(u) = \frac{1}{\cosh u}(u \cosh u - \sinh u,\ 1).$$

   Show that $\underline{\alpha}(u)$ is the curve followed by a stone starting at (0,1) on the end of a piece of rope of length 1 when the tractor on the other end of the piece of rope drives along the positive x-axis starting at $(0, 0)$, i. e. show that $\underline{\alpha}(u) + \underline{t}_{\underline{\alpha}}(u)$ is on the positive x-axis for $u > 0$ (and that $\underline{\alpha}(0) = (0, 1)$). Sketch the curve for all real values of u.

5. If $\underline{\alpha}$ denotes the catenary in Q.3, show that
   (i) the involute of $\underline{\alpha}$ starting from $(0, 1)$ is the tractrix in Q.4;
   (ii) the evolute of $\underline{\alpha}$ is the curve given by

$$\beta(t) = (t - \sinh t \cosh t, 2\cosh t).$$

   (iii) Find the singular points of β and give a sketch of its trace.

6. *Parallels.* Let $\underline{\alpha}$ be a plane curve parametrised by arc length and d a fixed real number. The curve $\underline{\beta}(u) = \alpha(u) + d\underline{n}(u)$ is called the parallel to $\underline{\alpha}$ at distance d.
   (i) Show that $\underline{\beta}$ is a regular curve except for values of u for which $\kappa(u) \neq 0$ and $d = 1/\kappa(u)$, where κ is the curvature of $\underline{\alpha}$.
   (ii) Show that the set of singular points of the parallels is the evolute of $\underline{\alpha}$.
   (iii) Use Maple (or any other software) to draw some parallels to the ellipse $\underline{\alpha}(t) = (2\cos(t), \sin(t))$. (The parametrisation is not arc length!). Draw the evolute of the ellipse. Does your figure agree with (i) and (ii)?

7. *Contact with circles.* A circle has equation $C(x, y) = 0$ where

$$C(x, y) = (x - a)^2 + (y - b)^2 - \lambda.$$

Let $\underline{\alpha} = (x(t), y(t))$ be a plane curve. Suppose that the point $\underline{\alpha}(t_0)$ is also on the circle, so the constant λ is so that C vanishes at $(x(t_0), y(t_0))$. Then the equation $g(t) = 0$ with

$$g(t) = C(x(t), y(t)) = (x(t) - a)^2 + (y(t) - b)^2 - \lambda$$

has a solution at t_0. If t_0 is a multiple solution of the equation, with $g^{(i)}(t_0) = 0$ for $i = 1, \ldots, k-1$ but $g^{(k)}(t_0) \neq 0$, we say that the curve $\underline{\alpha}$ and the circle have k-point contact at $\underline{\alpha}(t_0)$.

(i) Show that $\underline{\alpha}$ and the circle have 2-point contact at $\underline{\alpha}(t_0)$ if the circle is tangent to $\underline{\alpha}$ at $\underline{\alpha}(t_0)$.

(ii) Suppose that $\kappa(t_0) \neq 0$. Show that $\underline{\alpha}$ and the circle have at least 3-point contact at $\underline{\alpha}(t_0)$ if and only if the centre of the circle is the centre of curvature of $\underline{\alpha}$ at $\underline{\alpha}(t_0)$.

(iii) Show that $\underline{\alpha}$ and the circle have at least 4-point contact if and only if the centre of the circle is the centre of curvature of $\underline{\alpha}$ at $\underline{\alpha}(t_0)$ and $\underline{\alpha}(t_0)$ is a vertex of $\underline{\alpha}$.

II. Space curves

8. Let $\underline{\alpha}(u)$ be a regular curve in $\mathbf{R}^3$. Show that the curvature κ and the torsion τ of $\underline{\alpha}$ are given by

$$\kappa = \frac{|\underline{\alpha}' \times \underline{\alpha}''|}{|\underline{\alpha}'|^3}, \qquad \tau = -\frac{(\underline{\alpha}' \times \underline{\alpha}'') \cdot \underline{\alpha}'''}{|\underline{\alpha}' \times \underline{\alpha}''|^2},$$

where $'$ denotes differentiation with respect to u.

9. Find the curvature and torsion of the curve

$$\underline{\alpha}(u) = (au, bu^2, cu^3).$$

10. Consider the regular curve given by

$$\underline{\alpha}(s) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ b\frac{s}{c}\right), \qquad s \in \mathbf{R}$$

where $a, b, c > 0$ and $c^2 = a^2 + b^2$. Then $\underline{\alpha}(s)$ is a *helix.*

(i) Show that the trace of $\underline{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(ii) Show that $\underline{\alpha}$ is parametrized by arc length.

(iii) Determine the curvature and torsion of $\underline{\alpha}$ (and notice that they are both constant).

(iv) Determine the osculating plane of $\underline{\alpha}$ at each point.

(v) Show that the line through $\underline{\alpha}(s)$ in direction $\underline{n}(s)$ meets the axis of the cylinder orthogonally.

(vi) Show that the tangent lines to $\underline{\alpha}$ make a constant angle with the axis of the cylinder.

In fact, a helix is defined to be any curve on any circular cylinder with property (vi); see next question.

11. A curve $\underline{\alpha} : I \to \mathbf{R^3}$ is called a helix if its tangent lines make a constant angle with a fixed direction in $\mathbf{R^3}$. Suppose that $\tau(s) \neq 0$ for all $s \in I$.
(a) Prove that $\underline{\alpha}$ is a helix if and only if $\kappa/\tau = constant$.
(b) Prove that the curve

$$\underline{\alpha}(s) = \left(\tfrac{a}{c}\int \sin\theta(s)ds,\ \tfrac{a}{c}\int \cos\theta(s)ds,\ \tfrac{b}{c}s\right),$$

with $c^2 = a^2 + b^2$, $a \neq 0$, $b \neq 0$ and $\theta'(s) > 0$ is a helix.

12. Let $\underline{\alpha}(u)$, $\underline{\beta}(u)$ be regular curves in $\mathbf{R^3}$ such that, for each u, the principal normals $\underline{n}_\alpha(u)$ and $\underline{n}_\beta(u)$ are parallel. Prove that the angle between $\underline{t}_\alpha(u)$ and $\underline{t}_\beta(u)$ is independent of u. Prove also that if the line through $\underline{\alpha}(u)$ in direction $\underline{n}_{\alpha(u)}$ is equal to the line through $\underline{\beta}(u)$ in direction $\underline{n}_\beta(u)$ then

$$\underline{\beta}(u) = \underline{\alpha}(u) + r\underline{n}_{\underline{\alpha}}(u)$$

for some constant real number r.

13. Let $\underline{\alpha}(u)$ be the curve in $\mathbf{R^3}$ given by

$$\underline{\alpha}(u) = e^u(\cos u, \sin u, 1), \qquad u \in \mathbf{R}.$$

If $0 < \lambda_0 < \lambda_1$, find the length of the segment of $\underline{\alpha}$ which lies between the planes $z = \lambda_0$ and $z = \lambda_1$. Show also that the curvature and torsion of $\underline{\alpha}$ are both inversely proportional to e^u. Finally, show that the involute of $\underline{\alpha}$, starting from any point of $\underline{\alpha}$, is a plane curve.

14. Let $\underline{\beta}$ be the involute emanating from $(a, 0, 0)$ of the helix in Q.10 defined by

$$\underline{\beta}(s) = \underline{\alpha}(s) - s\underline{t}_{\underline{\alpha}}(s), \qquad s > 0.$$

Prove that $\underline{\beta}$ lies in the plane $z = 0$ (why is this lucky for maypole dancers?) and is the involute emanating from $(a, 0, 0)$ of the circle of intersection of the plane $z = 0$ with the cylinder $x^2 + y^2 = a^2$.

15. Let $\underline{\alpha}(s)$ be a curve parametrized by arc length with nowhere vanishing curvature and torsion. Show that $\underline{\alpha}$ lies on a sphere if and only if

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{1}{\tau\kappa^2}\frac{d\kappa}{ds}\right).$$

16. Let $\underline{\alpha}$ be a regular curve parametrised by arc length with $\kappa > 0$ and $\tau \neq 0$.
(a) If $\underline{\alpha}$ lies on a sphere of centre $\underline{c}$ and radius r, show that

$$\underline{\alpha} - \underline{c} = -\rho\underline{n} - \rho'\sigma\underline{b},$$

where $\rho = \frac{1}{\kappa}$ and $\sigma = -\frac{1}{\tau}$. Deduce that $r^2 = \rho^2 + (\rho'\sigma)^2$.
(b) Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that $\underline{\alpha}$ lies on a sphere of radius r. (Hint: show that the curve $\underline{\alpha} + \rho\underline{n} + \rho'\sigma\underline{b}$ is constant.)

III. Surfaces

17. Let $S^2(1) = \{(x,y,z) \in \mathbf{R^3} | \, x^2+y^2+z^2 = 1\}$. For $(u,v) \in \mathbf{R^2}$, let $\underline{x}(u,v)$ be the point of intersection of the line in $\mathbf{R^3}$ through $(u,v,0)$ and $(0,0,1)$ with $S^2(1)$. Find an explicit formula for $\underline{x}(u,v)$. Show that $\underline{x}$ is a local parametrization of $S^2(1)$ which covers $S^2(1) \setminus \{(0,0,1)\}$.

18. Show that each of the following is a surface:
    (i) the cylinder $\{(x,y,z) \in \mathbf{R^3} | \, x^2+y^2 = 1\}$;
    (ii) the hyperboloid of two sheets given by $\{(x,y,z) \in \mathbf{R^3} | \, x^2+y^2 = z^2-1\}$.
    In each case find a covering of the surface by coordinate neighbourhoods and give a sketch of the surface indicating the coordinate neighbourhoods you have used.

19. Let $f(x,y,z) = (x+y+z-1)^2$.
    (i) Find the points at which $\operatorname{grad} f = 0$.
    (ii) For which values of c is the set determined by the equation $f(x) = c$ a surface?
    (iii) What is the set determined by the equation $f(x) = c$?
    (iv) Repeat (i) and (ii) using the function $f(x,y,z) = xyz^2$.

20. Let $S = \{(x,y,z) \in \mathbf{R^3} | \, z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\}$. Show that S is a surface and show that at each point $p \in S$ there are two straight lines passing through p and lying in S (i. e. S is a *doubly ruled* surface).

21. Let $\underline{x}(u,v) = \underline{\alpha}(v) + u\underline{w}(v)$ be a parametrization of a ruled surface S such that $|\underline{w}(v)| \equiv 1$. A curve $\underline{\beta}(v)$ lying in S is called a *curve of striction* if $\underline{\beta}'.\underline{w}' \equiv 0$. Find the curve of striction of the ruled surface in the previous question (using either one of the rulings). (Hint: you may assume $\underline{\beta}(v) = \underline{\alpha}(v) + u(v)\underline{w}(v)$.)

22. Show that the equation $xz + y^2 = 1$ defines a surface S in $\mathbf{R^3}$. If $\underline{\alpha}(v) = (\cos v, \sin v, \cos v)$ and $\underline{\beta}(v) = (1+\sin v, -\cos v, -1+\sin v)$ show that, for all $v \in \mathbf{R}$, there are two straight lines through $\underline{\alpha}(v)$, one of which is in direction $\underline{\beta}(v)$, both of which lie on S. If $\underline{x}(u,v) = \underline{\alpha}(v) + u\underline{\beta}(v)$, $u \in \mathbf{R}$, $0 < v < 2\pi$, show that $\underline{x}$ is a local parametrization of S.

23. Determine all surfaces of revolution which are also ruled surfaces.

24. If a,b,$c > 0$ show that the ellipsoid S in $\mathbf{R^3}$ defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a surface and that

$$\underline{x}(u,v) = (a \sin u \cos v, b \sin u \sin v, c \cos u), \qquad 0 < u < \pi, \; 0 < v < 2\pi$$

is a local parametrization of S.

25. Let S be the surface in $\mathbf{R^3}$ defined by $z = x^2 - y^2$. If

$$\underline{x}(u,v) = (u + \cosh v, u + \sinh v, 1 + 2u(\cosh v - \sinh v)), \qquad u, v \in \mathbf{R},$$

show that $\underline{x}$ is a local parametrization of S.

26. (Moebius band) Let S be the image of the function $f : \mathbf{R} \times (-\epsilon, \epsilon) \to \mathbf{R^3}, (\epsilon > 0)$, defined by

$$f(u,v) = ((2 - v\sin\frac{u}{2})\sin u,\ (2 - v\sin\frac{u}{2})\cos u,\ v\cos\frac{u}{2}).$$

Show that, for ϵ sufficiently small, S is a surface in $\mathbf{R^3}$ which may be covered by two coordinate neighbourhoods. Give a sketch of the surface indicating the coordinate curves.

27. (Real projective plane) Let $f : \mathbf{R^3} \to \mathbf{R^5}$ be defined by

$$f(x,y,z) = (yz, zx, xy, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2)).$$

Show that:
(i) $f(x,y,z) = f(x',y',z')$ if and only if $(x,y,z) = \pm(x',y',z')$;
(ii) the image $S = f(S^2(1))$ of the unit sphere $S^2(1)$ in $\mathbf{R^3}$ is a surface in $\mathbf{R^5}$. (S is often written as $\mathbf{R}P^2$ and is called the *real projective plane*. Note that it can be identified with the set of lines through the origin in $\mathbf{R^3}$).
(Hint: For (ii) you may find it helpful to consider the open subsets W_x, W_y, W_z of $\mathbf{R^5}$ given by

$$W_x = \{(x_1,x_2,x_3,x_4,x_5) | x_4 + \frac{1}{\sqrt{3}}x_5 + \frac{1}{3} > 0\},$$

$$W_y = \{(x_1,x_2,x_3,x_4,x_5) | -x_4 + \frac{1}{\sqrt{3}}x_5 + \frac{1}{3} > 0\},$$

$$W_z = \{(x_1,x_2,x_3,x_4,x_5) | x_5 < \frac{1}{2\sqrt{3}}\}.$$

and use the fact that the intersections of S with W_x, W_y and W_z are the images of the hemispheres of $S^2(1)$ given by $x > 0, y > 0$ and $z > 0$ respectively).

28. Find the coefficients of the first fundamental form of $S^2(1)$ with respect to the local parametrization $\underline{x}$ defined in Q17.

29. Find the coefficients of the first fundamental forms of:
(i) the catenoid parametrized by

$$\underline{x}(u,v) = (\cosh v\cos u, \cosh v\sin u, v), \qquad (u,v) \in (0, 2\pi) \times \mathbf{R};$$

(ii) the helicoid parametrized by

$$\underline{\tilde{x}}(u,v) = (-\sinh v \sin u, \sinh v \cos u, -u), \qquad (u,v) \in (0,2\pi) \times \mathbf{R}.$$

(iii) the surface S_θ (θ constant) parametrized by

$$\underline{y}_\theta(u,v) = \cos\theta\, \underline{x}(u,v) + \sin\theta\, \underline{\tilde{x}}(u,v), \qquad (u,v) \in (0,2\pi) \times \mathbf{R}.$$

30. Let $\underline{x}(u,v)$ be a local parametrization of a surface S. Show that, in the usual notation, the vector $\alpha\underline{x}_u + \beta\underline{x}_v$ bisects the angle between the coordinate curves if and only if

$$\sqrt{G}(\alpha E + \beta F) = \sqrt{E}(\alpha F + \beta G)\ .$$

If

$$\underline{x}(u,v) = (u, v, u^2 - v^2)\ ,$$

find a vector tangential to S which bisects the angle between the coordinate curves at the point $(1,1,0)$.

31.

(i) A local parametrization $\underline{x}$ of a surface S in $\mathbf{R^3}$ is orthogonal provided $F = 0$ (so $\underline{x}_u$ and $\underline{x}_v$ are orthogonal at each point). Show that, in this case, at any point $p = \underline{x}(u,v)$ on S,

$$-d\underline{\mathrm{N}}_p(\underline{x}_u) = \tfrac{L}{E}\underline{x}_u + \tfrac{M}{G}\underline{x}_v$$

$$-d\underline{\mathrm{N}}_p(\underline{x}_v) = \tfrac{M}{E}\underline{x}_u + \tfrac{N}{G}\underline{x}_v$$

where $\underline{\mathrm{N}}$ denotes the Gauss map and E, F, G (resp. L, M, N) are the coefficients of the first (resp. second) fundamental form.

(ii) A local parametrization $\underline{x}$ of a surface S in $\mathbf{R^3}$ is principal provided $F = M = 0$. Prove that, in this case, $\underline{x}_u$ and $\underline{x}_v$ are principal vectors at each point with corresponding principal curvatures L/E and N/G.

32. Let $f : \mathbf{R^4} \to \mathbf{R^2}$ be given by

$$f(x_1, x_2, x_3, x_4) = ({x_1}^2 + {x_2}^2, {x_3}^2 + {x_4}^2).$$

For each pair of positive real numbers r_1, r_2 show that $(r_1^2, r_2^2) \in \mathbf{R^2}$ is a regular value of f. Let $S = f^{-1}(r_1^2, r_2^2)$, and let

$$\underline{x}(u,v) = (r_1 \cos(u/r_1), r_1 \sin(u/r_1), r_2 \cos(v/r_2), r_2 \sin(v/r_2)),$$

$$0 < u < 2\pi r_1,\ 0 < v < 2\pi r_2.$$

Show that $\underline{x}(u,v)$ is a local parametrization of S and compute the coefficients of the first fundamental form of S with respect to this local parametrization.

33. Let $f(z)$ be a complex analytic function of the complex variable z. If $\mathbf{C}^2$ is identified with $\mathbf{R^4}$ in the usual way then the graph of f is a surface in $\mathbf{R^4}$ which is parametrised by $\underline{x}(z) = (z, f(z))$. Show that $\underline{x}(z)$ is an isothermal parametrization.

34. Let S be the subset of $\mathbf{R}^3$ given by

$$S = \left\{ \left(u, v, \frac{1}{2}(u^2 - v^2)\right) \mid (u, v) \in \mathbf{R}^2 \right\}.$$

(a) Show that S is a surface in $\mathbf{R}^3$ and that

$$\underline{x}(u, v) = (u, v, \frac{1}{2}(u^2 - v^2))$$

is a parametrization which covers S.

(b) Find the coefficients of the first fundamental form of S with respect to this parametrization.

(c) Find the angles between the coordinate curves at the points $(1, 0, 1/2)$ and $(1, 1, 0)$.

35. (a) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be defined by

$$f(x, y, z) = x \sin z - y \cos z.$$

Find the points at which grad f vanishes, and hence show that the set S with equation $f(x, y, z) = 0$ is a surface.

(b) Show that $\underline{x}(u, v) = (\sinh v \cos u, \sinh v \sin u, u)$ is a parametrization which covers the whole of S.

(c) Find the coefficients of the first fundamental form of S with respect to this parametrization.

36. Let (u, v), $u \in \mathbf{R}, v > 0$, be a parametrization of a surface $\mathbf{H}$ in $\mathbf{R^2}$ with $E = G = 1/v^2$, $F = 0$. Then $\mathbf{H}$ is the *hyperbolic plane*. Let $c > 0$ and let $\underline{\alpha}(t) = (c \cos t, c \sin t)$, $\quad \pi/6 \leq t \leq 5\pi/6$. Show that the length of $\underline{\alpha}$ in $\mathbf{H}$ is equal to $\int_{\pi/6}^{5\pi/6} \frac{1}{\sin t} dt$. (In fact $\underline{\alpha}$ is the curve of shortest length between its endpoints.) Now take $c = \sqrt{2}$ and find the angle of intersection of $\underline{\alpha}$ with the curve $\underline{\beta}(s) = (1, s)$ at their point of intersection.

37. Find two families of curves on the helicoid which, at each point, bisect the angles between the coordinate curves of the parametrisation given by

$$\underline{x}(u, v) = (v \cos u, v \sin u, u).$$

(Show that they are given by $u \pm \sinh^{-1} v = c$, where c is a constant on each curve in the family.)

38. A system of curves on the cylinder $x^2 + (y-a)^2 = a^2$ $(a \neq 0)$ is given by the intersection of the cylinder with the paraboloids $xz = \lambda y$, where λ is a parameter. Show that every orthogonal trajectory of this system lies on a sphere with fixed centre.

39. Let $\underline{x}(u,v) = (v\cos u, v\sin u, v + u\sqrt{2})$ be a parametrization of a surface S.
    (i) Find the orthogonal trajectories in S to the family of curves $\mathcal{F}$ obtained by intersecting S with the planes $z =$ constant.
    (ii) Find the angle of intersection at the point $(\sqrt{2}, 0, \sqrt{2})$ of the coordinate curve $v =$ constant with the curve in the family $\mathcal{F}$.
    (iii) Find a family of curves in S which bisects the angles between the coordinate curves at each point.

40. Show that if all the normals to a path-connected surface pass through a given point then the surface is contained in a sphere. (You will need a bit of elementary topology to do this).

41. Find a basis of the tangent plane at the point $(a/2, b/2, c/\sqrt{2})$ to the ellipsoid S of Q24. If $\underline{x}$ is the local parametrization introduced in Q24 and $L(v)$ is the length of the regular curve $\underline{\alpha}(t) = \underline{x}(t,v)$, $0 < t < \pi$, show that L has stationary values at $v = \pi/2$, π, $3\pi/2$, and interpret this fact geometrically.

42. Let S be a surface parametrized by

$$\underline{x}(u,v) = (u\cos v, u\sin v, \log\cos v + u), \qquad -\frac{\pi}{2} < v < \frac{\pi}{2}, u \in \mathbf{R}.$$

    For $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$, let $\underline{\alpha}_c$ be the coordinate curve $v = c$. Show that the length of $\underline{\alpha}_c$ from $u = u_0$ to $u = u_1$ is independent of c.

43. Let $f : S^2(1) \to S^2(1)$ be given by $f(p) = -p$. Show that f is a diffeomorphism. (The map f is called the *antipodal map*.)

44. Construct a diffeomorphism from the ellipsoid of Q24 to $S^2(1)$.

45. Let S be a surface in $\mathbf{R^n}$ and let $\underline{v}$ be a unit vector in $\mathbf{R^n}$. Let $h : S \to \mathbf{R}$ be given by $h(p) = p.\underline{v}$. Show that h is smooth and that $p \in S$ is a critical point of h if and only if $\underline{v}$ is normal to S at p.

46. Construct a local isometry from the plane $P = \{(x,y,0) \in \mathbf{R^3} | \, x, y \in \mathbf{R}\}$ onto the cylinder $x^2 + y^2 = a^2$, $(a \neq 0)$.

47. Let b be a positive number such that $\sqrt{1+b^2}$ is an integer n. Let S be the circular cone obtained by rotating the curve $\underline{\alpha}(v) = (v, 0, bv)$, $(v > 0)$, about the z-axis. If P is the plane defined in the previous question, show that the map $f : P \setminus \{(0,0,0)\} \to S$ given by

$$f(r\cos\theta, r\sin\theta, 0) = \frac{1}{n}(r\cos n\theta, r\sin n\theta, br)$$

    is a local isometry. Make a model to illustrate the map f using a sheet of paper.

48. Let S be the surface defined in Q32 and let S_1 be the cylinder in $\mathbf{R^3}$ given by

$$S_1 = \{(x, y, z) \in \mathbf{R^3} |\, x^2 + y^2 = r_1^2\}.$$

Let $g : S_1 \to \mathbf{R^4}$ be given by

$$g(x, y, z) = (x, y, r_2 \cos(z/r_2), r_2 \sin(z/r_2)).$$

Show that g defines a surjective local isometry from S_1 to S.

49. Let $f : S^2(1) \to S^2(1)$ be the map defined by

$$f(p) = {\pi_N}^{-1} T \pi_N(p) \quad \text{if} \quad p \neq (0,0,1), \qquad\qquad f(p) = p \quad \text{if} \quad p = (0,0,1),$$

where $\pi_N : S^2(1) \setminus \{(0,0,1)\} \to \mathbf{C}$ is the diffeomorphism defined by $\pi_N(x, y, z) = \dfrac{x + iy}{1 - z}$ and $T : \mathbf{C} \to \mathbf{C}$ is the function defined by $T(w) = aw + b$, where $a, b \in \mathbf{C}, a \neq 0$. Show that f is a conformal diffeomorphism of $S^2(1)$. (In particular this involves showing that f is smooth and conformal at $(0, 0, 1)$).

Give a sketch of $S^2(1)$ showing the curves of intersection of $S^2(1)$ with the coordinate planes and their images under f when $a = b = 1$.

50. Show that the map $f : S^2(1) \to S^2(1)$ defined in the previous question is an isometry if and only if $|a| = 1$ and $b = 0$. Hence show that every isometry of $S^2(1)$ which fixes $(0, 0, 1)$ is a rotation about the z-axis.

51. Suppose that $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are local isometries between surfaces. Prove that $f_2 \circ f_1 : S_1 \to S_3$ is a local isometry.

52. Let $f : S^3(1) \setminus \{(0,0,0,1)\} \to \mathbf{R^3}$ be defined by

$$f(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1 - x_4}, \frac{x_2}{1 - x_4}, \frac{x_3}{1 - x_4} \right),$$

(so that f is just stereographic projection from $(0, 0, 0, 1)$ to the plane $x_4 = 0$, compare with your solution to Q17). Show that f defines a conformal diffeomorphism of the torus in $\mathbf{R^4}$

$$S^1(r_1) \times S^1(r_2) = \{(x_1, x_2, x_3, x_4) \in \mathbf{R^4} |\, {x_1}^2 + {x_2}^2 = {r_1}^2, {x_3}^2 + {x_4}^2 = {r_2}^2\} ,$$

with ${r_1}^2 + {r_2}^2 = 1$, onto the torus of revolution $T_{a,b}$ in $\mathbf{R^3}$ obtained by rotating the circle

$$(x - a)^2 + z^2 = b^2 , \quad y = 0$$

about the z-axis, where $a = 1/r_1$ and $b = r_2/r_1$.

53. Let $H = \{(u, v) \in \mathbf{R^2} | \, v > 0\}$ be the hyperbolic plane with metric

$$ds^2 = \frac{1}{v^2}(du^2 + dv^2),$$

and let S be the pseudosphere obtained by rotating the tractrix

$$\underline{\alpha}(t) = (\frac{1}{\cosh t}, \ 0, \ t - \tanh t) \quad , \qquad\qquad t \geq 0,$$

around the z-axis. Show that, with respect to a suitable choice of parametrization, the metric on S is given by

$$ds^2 = \frac{1}{\cosh^2 t}(du^2 + \sinh^2 t dt^2),$$

and, by considering the change of variable $v = \cosh t$, show that there is a local isometry of the open subset $H' = \{(u, v) \in H | \, v > 1\}$ of H onto S.

54. Let S be the surface of revolution parametrized by

$$\underline{x}(u, v) = (\cos v \cos u, \cos v \sin u, -\sin v + \log \tan(\frac{\pi}{4} + \frac{v}{2})),$$

where $0 < u < 2\pi, 0 < v < \pi/2$. Let S_1 be the region

$$S_1 = \{\underline{x}(u, v) | \, 0 < u < \pi, 0 < v < \frac{\pi}{2}\}$$

and let S_2 be the region

$$S_2 = \{\underline{x}(u, v) | \, 0 < u < 2\pi, \frac{\pi}{3} < v < \frac{\pi}{2}\}.$$

Show that the map

$$\underline{x}(u, v) \mapsto \underline{x}(2u, \arccos(\frac{1}{2}\cos v))$$

is an isometry from S_1 onto S_2.

55. Let S be a surface of revolution. Prove that any rotation about the axis of revolution is an isometry of S.

56. (The disk model of the hyperbolic plane). Let $\tilde{H}$ denote the unit disk with the metric

$$ds^2 = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2},$$

and let H be the hyperbolic plane of Q53. Show that the map $f : H \to \tilde{H}$ given by

$$f(w) = \frac{w - i}{w + i}, \qquad\qquad w = u + iv \in H,$$

is an isometry.

57. (The hyperbolic plane for relativity theorists!). Let $Q : \mathbf{R^3} \to \mathbf{R}$ be the quadratic form defined by

$$Q(x, y, z) = x^2 + y^2 - z^2, \qquad (x, y, z) \in \mathbf{R^3}.$$

(This is sometimes called an indefinite metric on $\mathbf{R^3}$). Let

$$S = \{(x, y, z) \in \mathbf{R^3} | Q(x, y, z) = -1\}.$$

(Thus S is a hyperboloid of two sheets, and is a " sphere of radius $\sqrt{-1}$" in terms of the indefinite metric). Show that the induced quadratic form on each tangent plane T_pS is positive definite and that the map $f : H \to S$ from the disk model of the hyperbolic plane (see Q56) defined by

$$f(u, v) = \frac{(2u, 2v, 1 + u^2 + v^2)}{1 - u^2 - v^2}, \qquad (u, v) \in H,$$

maps H isometrically onto the component of S for which $z > 0$.

58. Let a,b,c, be non-zero real numbers. Show that each of the equations

$$x^2 + y^2 + z^2 = ax,$$

$$x^2 + y^2 + z^2 = by,$$

$$x^2 + y^2 + z^2 = cz,$$

defines a surface and that each pair of surfaces intersects orthogonally at all points of intersection. (Note, incidentally, that each of these surfaces is a sphere.)

59. Determine the Gauss map for both the catenoid $x^2+y^2 = \cosh^2 z$ and the helicoid $x \sin z = y \cos z$ and show that in each case it is conformal. (You will find it both helpful and interesting to use the isothermal coordinates of Q29). Show that the image of the Gauss map is the subset $S^2(1) \setminus \{(0, 0, \pm 1)\}$ of $S^2(1)$ in each case and prove that the Gauss map is injective for the catenoid but not the helicoid. (You might also find it interesting to consider the Gauss map for the surfaces S_θ in Q29(iii)).