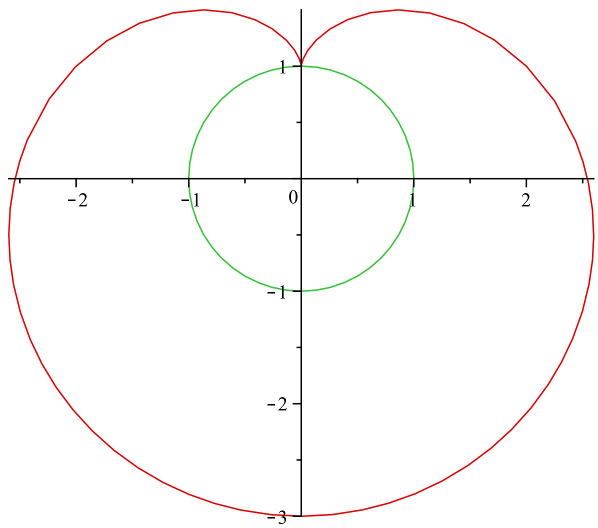
Differential Geometry III (Math 3021)

Solutions to the First Homework 1.11.2012

Question 2 The graph of the epicycloid is illustrated below for the value $r = 1$.



Note that the inner circle with centre $(0, 0)$ is fixed and a second circle rotates around it with a marked point on its perimeter tracing out the epicycliod. This point is at the bottom of the rotating circle at the moment when the rotating circle is just on top of the fixed circle, i.e., at position $(0, r)$. Assume that the mid point of the rotating circle is located at $(2r \sin u, 2r \cos u)$. At $u = 0$ this means the mid point of the rotating circle is at $(0, 2r)$ and the marked point is at $(r \sin \pi, r \cos \pi) = (0, -r)$ seen from this mid point. As u increases, the mid point moves clockwise around the origin, and so does the point of contact between the fixed and the rotating circle, and also so does the marked point around this mid point in relation to the point of contact. Once the mid point is at $(2r \sin u, 2r \cos u)$, the rotating circle has rotated clockwise around its moving centre by a total length of $2ru$, where u is measured in radiants. Therefore the point of contact between the two circles, seen from the moving midpoint of the rotating circle, has moved clockwise by the angle u around its moving centre, and the position of the point of contact relative to this moving mid point is $(r \sin(\pi + u), r \cos(\pi + u))$. The marked point has moved clockwise away from the point of contact by the same angle, and is therefore at position $(r \sin(\pi + 2u), r \cos(\pi + 2u))$

relative to the centre of the moving circle. This means that the marked point lies at

$$(2r\sin u, 2r\cos u) + (r\sin(\pi + 2u), r\cos(\pi + 2u)) = \\ (2r\sin u - r\sin 2u, 2r\cos u - r\cos 2u).$$

Now let

$$\alpha(u) = (2r\sin u - r\sin 2u, 2r\cos u - r\cos 2u).$$

Then

$$\begin{aligned}
\alpha'(u) &= 2r(\cos u - \cos 2u, -(\sin u - \sin 2u)) \\
\|\alpha'(u)\|^2 &= 4r^2(2 - 2(\cos(-u)\cos(2u) - \sin(-u)\sin(2u)) \\
&= 4r^2(2 - 2\cos(2u - u)) = 4r^2(2 - 2\cos u) \\
&= 4r^2(2 - 2(\cos(u/2)\cos(u/2) - \sin(u/2)\sin(u/2))) \\
&= 16r^2\sin^2(u/2).
\end{aligned}$$

This implies that $\|\alpha'(u)\| = 4r\sin(u/2)$ and

$$l(\alpha) = \int_0^{2\pi} \|\alpha'(u)\| du = 4r\int_0^{2\pi} \sin\frac{u}{2} du = 4r\left(-2\cos\frac{u}{2}\Big|_0^{2\pi}\right) = -8r(\cos\pi - \cos 0) = 16r.$$

Question 6 (i) Assume $\kappa(u) = 0$ or $d\kappa(u) \neq 1$. The latter is automatically satisfies if $\kappa(u) = 0$. So we just assume that $d\kappa(u) \neq 1$. We need to show that $\beta'(u) \neq 0$. Since α is unit length, we have

$$\beta'(u) = t(u) + dn'(u) = t(u) + dt'(u)A = t(u) + d\kappa(u)n(u)A = \\ t(u) + d\kappa(u)t(u)A^2 = t(u) - d\kappa(u)t(u) = (1 - d\kappa(u))t(u),$$

with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that $\|t(u)\| = 1$, i.e., $t(u) \neq 0$. The initial assumption implies that $(1 - d\kappa(u)) \neq 0$ and, therefore $\beta'(u) \neq 0$, i.e., $\beta(u)$ is regular.

In the case $\kappa(u) \neq 0$ and $d\kappa(u) = 1$, i.e., $d = 1/\kappa(u)$, we obviously have $\beta'(u) = 0$, i.e., $\beta(u)$ is singular.

(ii) The evolute is only defined in the case that we have $\kappa(u) \neq 0$ for all u. So we assume this. We have seen in (i) that the singular points of the parallels are precisely those $\beta(u)$ for which we have $d\kappa(u) = 1$, i.e., $d = 1/\kappa(u)$. This means that

$$\beta(u) = \alpha(u) + dn(u) = \alpha(u) + \frac{1}{\kappa(u)}n(u),$$

which is precisely the parametrisation of the evolute of α.

Question 10 (i) Let

$$\alpha(s) = (x(s), y(s), z(s)) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \quad s \in \mathbb{R},$$

with $c^2 = a^2 + b^2$. Then

$$x(s)^2 + y(s)^2 = a^2 \left(\cos^2 \frac{s}{c} + \sin^2 \frac{s}{c}\right) = a^2,$$

i.e., the trace of α lies on the cylinder $x^2 + y^2 = a^2$.

(ii) We have

$$\alpha'(s) = \left(-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c}\right),$$

which implies

$$\|\alpha'(s)\|^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1.$$

This shows that α is unit speed.

(iii) We have

$$\begin{aligned} \alpha''(s) &= \left(-\frac{a}{c^2} \cos \frac{s}{c}, -\frac{a}{c^2} \sin \frac{s}{c}, 0\right), \\ \alpha'''(s) &= \left(\frac{a}{c^3} \sin \frac{s}{c}, -\frac{a}{c^3} \cos \frac{s}{c}, 0\right). \end{aligned}$$

This implies

$$\alpha'(s) \times \alpha''(s) = \begin{pmatrix} \begin{vmatrix} \frac{a}{c} \cos \frac{s}{c} & -\frac{a}{c^2} \sin \frac{s}{c} \\ \frac{b}{c} & 0 \end{vmatrix} \\ -\begin{vmatrix} -\frac{a}{c} \sin \frac{r}{c} & -\frac{a}{c^2} \cos \frac{s}{c} \\ \frac{b}{c} & 0 \end{vmatrix} \\ \begin{vmatrix} -\frac{a}{c} \sin \frac{s}{c} & -\frac{a}{c^2} \cos \frac{s}{c} \\ \frac{a}{c} \cos \frac{s}{c} & -\frac{a}{c^2} \sin \frac{s}{c} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} \frac{ab}{c^3} \sin \frac{s}{c} \\ -\frac{ab}{c^3} \cos \frac{s}{c} \\ \frac{a^2}{c^3} \end{pmatrix}.$$

We conclude that

$$\|\alpha'(s) \times \alpha''(s)\|^2 = \frac{a^2(a^2 + b^2)}{c^2} = \frac{a^2}{c^4},$$

i.e.,

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{a}{c^2}.$$

Moreover, we have

$$(\alpha'(s) \times \alpha''(s)) \cdot \alpha'''(s) = \frac{a^2 b}{c^6} \sin^2 \frac{s}{c} + \frac{a^2 b}{c^6} \cos^2 \frac{s}{c} = \frac{a^2 b}{c^6}.$$

This implies that

$$\tau = -\frac{a^2 b}{c^6} \cdot \frac{c^4}{a^2} = -\frac{b}{c^2}.$$