On conjugacy growth in groups

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Newcastle, June 6, 2016

1. Counting conjugacy classes

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- Conjugacy growth of G: number of conjugacy classes intersecting the ball (or sphere) of radius n in the Cayley graph of G w.r.t. X.

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- The strict conjugacy growth function is then

$$c_{G,X}(n) := \sharp\{[g] \in G \mid |g|_c = n\}$$

and the cumulative one is

$$cc_{G,X}(n) := \sharp\{[g] \in G \mid |g|_c \leq n\}$$

Conjugacy vs. standard growth

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Quasi-isometry invariant	yes	no**
Rate of growth		

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** Hull-Osin (2013): conjugacy growth not quasi-isometry invariant.

Conjugacy growth in geometry

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A slight modification of the conjugacy growth function (including only the non-powers) appears in geometry:

- counting the primitive closed geodesics of bounded length on a compact manifold M of negative curvature and exponential volume growth gives, via quasi-isometries, good (exponential) asymptotics for the conjugacy growth of the fundamental group of M (Margulis, ...).

Conjugacy growth asymptotics

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- **Conjecture (Guba-Sapir)**: most groups of standard exponential growth should have exponential conjugacy growth. Exclude the Osin or Ivanov type 'monsters'!

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- **Breuillard-Cornulier-Lubotzky-Meiri (2011)**: uniform exponential conjugacy growth for f.g. linear (non virt. nilpotent) groups.
- **Hull-Osin (2014)**: all acylindrically hyperbolic groups have exponential conjugacy growth.

Growth rates

Let $a(n) = |S_X(n)|$ be the number of elements of length n in G wrt X.

The standard growth rate of G wrt X is

$$\alpha = \alpha_{G,X} = \limsup_{n\to\infty} \sqrt[n]{a(n)}.$$

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Since $a(n+m) \leq a(n)a(m)$, we have (Fekete's Lemma)

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{a(n)} = \lim_{n\to\infty} \sqrt[n]{a(n)} = \inf_n \sqrt[n]{a(n)}.$$

Conjugacy growth rates

Let $c(n)$ be the number of conjugacy classes of length n in G wrt X.

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Question (Breuillard, Cornulier, Lubotzky, Meiri):

$$\liminf_{n\to\infty} \sqrt[n]{c(n)} \leq \limsup_{n\to\infty} \sqrt[n]{c(n)} \leq \lim_{n\to\infty} \sqrt[n]{a(n)}$$

Can the first inequality be strict?

Conjugacy vs. standard growth

	Standard growth	Conjugacy growth
Type	pol., int., exp.	pol., int., exp.
Quasi-isometry invariant	yes	no
Rate of growth	limit exists	???

Growth rates from power series

Let $(a_i)_{i\geq 0}$ be a sequence of integers and $f(z) = \sum_{i=0}^{\infty} a_i z^i$ be a complex power series. The radius of convergence of f is

$$RC(f) = \sup\{r \in \mathbb{R} \mid f(z) \text{ converges } \forall\, z \in D(0,r)\},$$

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and

$$RC(f) = \frac{1}{\limsup_{i\to\infty} \sqrt[i]{a_i}} = \frac{1}{\alpha},$$

so one can determine exponential growth rate of the sequence $(a_i)_{i\geq 0}$ via the radius of convergence of its formal power series.

Radius of convergence for rational series

For any rational function $f(z) = \frac{P(z)}{Q(z)}$ the radius of convergence $RC(f)$ of f is the smallest absolute value of a pole of f, i.e. the smallest absolute value of a zero of $Q(z)$.

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Question: When are conjugacy growth series for groups rational?

The conjugacy growth series

Let G be a group with finite generating set X.

- The conjugacy growth series of G with respect to X records the number of conjugacy classes of every length. It is

$$\sigma_{(G,X)}(z) := \sum_{n=0}^{\infty} c_{(G,X)}(n) z^n,$$

where $c(n) = c_{(G,X)}(n)$ is the number of conjugacy classes of length n.

2. Conjugacy growth for

- Hyperbolic groups
- Graph products
- Generalized Baumslag-Solitar groups
- Wreath products

Hyperbolic groups

Asymptotics of conjugacy growth in the free group F_r

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Coornaert (2005): For the free group F_k, the primitive (non-powers) conjugacy growth function is given by

$$c_p(n) \sim \frac{(2r-1)^{n+1}}{2(r-1)n} = K\frac{(2r-1)^n}{n},$$

where $K = \frac{2r-1}{2(r-1)}$.

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In general, when powers are included, one cannot divide by n.

Asymptotics of conjugacy growth in hyperbolic groups

Theorem. (Coornaert - Knieper, Antolín - C.)

Let G be a non-elementary word hyperbolic group. Then there are positive constants A, B and n_0 such that

$$A\frac{\alpha^n}{n} \leq cc(n) \leq B\frac{\alpha^n}{n}$$

for all $n \geq n_0$, where α is the growth rate of G.

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MESSAGE:.

The number of conjugacy classes in the ball of radius n is asymptotically the number of elements in the ball of radius n divided by n.

Bounds for the conjugacy growth

Let $c_p(n) := \sharp\{\text{primitive } [g] \in G \mid |g|_c \leq n\}$ be the primitive cumulative conjugacy growth.

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Theorem (Coornaert and Knieper, IJAC 2004)

Let G be a torsion-free non-elementary word hyperbolic group. Then there are positive constants B and n_1 such that for all $n \geq n_1$

$$c_p(n) \leq B\frac{\alpha^n}{n}.$$

Conjugacy growth for all hyperbolic groups (Antolín-C.)

1. Allow torsion and modify the upper bound of Coornaert and Knieper:

   (i) use the fact that there exists $m < \infty$ such that all finite subgroups $F \leq G$ satisfy $|F| \leq m$.

   (ii) most ($\geq \frac{n}{m}$) cyclic permutations of a primitive conjugacy representative of length n correspond to different elements of length n in G.

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   (ii) most ($\geq \frac{n}{m}$) cyclic permutations of a primitive conjugacy representative of length n correspond to different elements of length n in G.

2. Find conjugacy growth upper bound for all conjugacy classes, i.e. include the non-primitive classes in the count.

■

Consequences

Corollary (AC)

For any hyperbolic group G with generating set X we have

$$\lim_{n\to\infty} \sqrt[n]{c(n)} = \gamma_{G,X} = \alpha_{G,X}.$$

Conjugacy growth series in virt. cyclic groups: $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

In $\mathbb{Z}$ the conjugacy growth series is the same as the standard one:

$$\sigma_{(\mathbb{Z},\{1,-1\})}(z) = 1 + 2z + 2z^2 + \cdots = \frac{1+z}{1-z}.$$

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ a set of conjugacy representatives is $1, a, b, ab, abab, \ldots$, so

$$\sigma_{(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1+2z-2z^3}{1-z^2}.$$

The conjugacy growth series in free groups

- Rivin (2000, 2010): the conjugacy growth series of F_k is not rational:

$$\sigma(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

This is combinatorics, not group theory!

Let L be a set of words, a_k the number of words of length k in L, and $f_L(t) = \sum_{k \geq 1} a_k t^k$ the generating function of L.

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Then the generating function for language $L/_\sim$ of cyclic representatives of L is

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$$\int_0^z \frac{\sum_{k \geq 1} \phi(k) f_L(t^k)}{t} \, dt,$$

and the growth rates of L and $L/_\sim$ are the same.

Rivin's formula for free groups

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Compare the general formula

$$\int_0^z \frac{\sum_{d\geq 1}\phi(d) f_L(t^d)}{t}\, dt$$

with Rivin's formula

$$\sigma(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty}\phi(d)\left(\frac{1}{1-(2k-1)x^d} - 1\right).$$

Conjecture (Rivin, 2000)

If G hyperbolic, then the conjugacy growth series of G is rational if and only if G is virtually cyclic.

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Theorem (Antolín-C., IMRN 2016)

If G is non-elementary hyperbolic, then the conjugacy growth series is transcendental.

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$\Leftarrow$

Theorem (C., Hermiller, Holt, Rees, IJM 2016)

Let G be a virtually cyclic group. Then the conjugacy growth series of G is rational.

NB: Both results hold for all symmetric generating sets of G.

Analytic combinatorics at work

The transcendence of the conjugacy growth series for non-elementary hyperbolic groups follows from the bounds

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Analytic combinatorics at work

The transcendence of the conjugacy growth series for non-elementary hyperbolic groups follows from the bounds

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

together with

Lemma (Flajolet: Trancendence of series based on bounds).

Suppose there are positive constants $A, B, \mathbf{h}$ and an integer $n_0 \geq 0$ s.t.

$$A\frac{e^{\mathbf{h}n}}{n} \leq a_n \leq B\frac{e^{\mathbf{h}n}}{n}$$

for all $n \geq n_0$. Then the power series $\sum_{i=0}^{\infty} a_n z^n$ is not algebraic.

Graph Products

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- Let $\Gamma = (V, E)$ be a simple graph with vertex set V and edge set E.
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- Let $\Gamma = (V, E)$ be a simple graph with vertex set V and edge set E.
- For each vertex v of Γ, let G_v be a group.
- The graph product of the groups G_v *with respect to* Γ is the quotient of their free product by the normal closure of the relators $[g_v, g_w]$ for all $g_v \in G_v$, $g_w \in G_w$ for which $\{v, w\} \in E$.

Note: indecomposable graph products are acylindrically hyperbolic.

(Minasyan-Osin)

Graph products

Theorem (C. - Mercier, 2016)

Let G be a graph product and assume that for each vertex group G_v the conjugacy and standard growth rates are the same, i.e. $\alpha_{G_v} = \gamma_{G_v}$.

Then the conjugacy and standard growth rates of G are the same, i.e.

$$\alpha_G = \gamma_G.$$

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Corollary. The conjugacy growth rate of a right-angled Artin/Coxeter group is the same as its standard growth rate.

Graph product decomposition

Let G be a graph product, and let $v \in V$. The group G can be decomposed as an amalgamated product as follows:

$$G_{V\setminus\{v\}} \underset{G_{N(v)}}{*} (G_{N(v)} \times G_v),$$

where $G_{N(v)} \hookrightarrow G_{V\setminus\{v\}}$ and $G_{N(v)} \hookrightarrow (G_{N(v)} \times G_v)$.

where $N(v)$ represents the neighbors of v and the two inclusions are admissible.

Graph product decomposition

Lemma (Lewin, Alonso (1991))

Let A, B and $C \leq A, B$ be groups with symmetric generating sets X, Y and Z, respectively. Assume that C is admissible in both A and B.

Let $G = A *_C B$ have generating set $W := X \cup Z$.

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Let $G = A *_C B$ have generating set $W := X \cup Z$. Then

$$\frac{1}{f_{(G,W)}} = \frac{1}{f_{(A,X)}} + \frac{1}{f_{(B,Y)}} - \frac{1}{f_{(C,Z)}},$$

where $f_{(G,W)}$ is the standard growth series of G wrt W.

Lemma on standard growth rates

Let G be a graph product, let $v \in V$ be a vertex, and let $A = G_{V \setminus \{v\}}$, $B = G_{N(v)}$, $C = G_{\{v\}}$. Then

$$G = A \underset{B}{*} (B \times C).$$

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Let U be the admissible right transversal of B in A: since $A = BU$, we have $f_A(z) = f_B(z) f_U(z)$. Also, B is admissible in $B \times C$ with transversal C, so

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$$f_G = \frac{f_B f_A f_C}{f_B f_C + f_A - f_A f_C} = f_B \frac{f_U f_C}{f_C + f_U - f_U f_C}.$$

In particular, $RC(f_G) = \inf\{|z| \,:\, f_C(z) + f_U(z) - f_U(z) f_C(z) = 0\}$.

Lemma (on conjugacy representatives)

Let $u_i \in U$ and $c_i \in C$ be nontrivial geodesics, $1 \leq i \leq n$. Then the elements

$$u_1 c_1 \cdots u_n c_n, \tag{1}$$

are of minimal length in their conjugacy class in G.

Moreover, two such elements are conjugate iff they are cyclically conjugate.

Lemma (growth rate of cyclic representatives)

The growth rate of the set of cyclic representatives of the set $u_1 c_1 \cdots u_n c_n$ is the smallest absolute value of $z \in \mathbb{C}$ such that

$$f_C(z) + f_U(z) - f_U(z) f_C(z) = 0,$$

which is the same as $RC(f_G) = \inf\{|z| \,:\, f_C(z) + f_U(z) - f_U(z) f_C(z) = 0\}$.

Groups acting on trees

(Super-)Generalized Baumslag-Solitar groups

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- A generalized Baumslag-Solitar (GBS) group is the fundamental group of a graph of groups with all vertex groups fg free abelian (of same rank). Equivalently, a GBS is a group which acts on a tree with all vertex and edge stabilizers fg free abelian.

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Examples:

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Examples:

1. $BS(n,m) = \langle a,t \mid ta^n t^{-1} = a^m \rangle$ are not acylindrically hyperbolic unless $m = 0$ or $n = 0$ (Osin).
2. $BS(1,n)$ is solvable, but $BS(2,3) = \langle a,t \mid ta^2 t^{-1} = a^3 \rangle$ is not.

Generalized Baumslag-Solitar groups

Theorem (C. - Coulon, 2016)

The fundamental group of a graph of groups with all vertex groups fg abelian has exponential conjugacy growth if it has exponential standard growth.

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Proposition

The conjugacy and standard growth rates are the same for $BS(1, p)$, where $p > 2$ is a prime.

Wreath products

Wreath products

Theorem (Mercier, 2016)

The conjugacy and standard growth rates are the same for groups of the form $G \wr L$, where G is any group and L is a group whose Cayley graph is a tree.

$*$ Also, explicit computations of conjugacy growth series.

Infinitely generated wreath products

Bacher - de la Harpe (2016)

Explicit computations of the conjugacy growth series of

- $Sym(\mathbb{N})$, the finitary symmetric group of the natural numbers,
- $Alt(\mathbb{N})$, the finitary alternating group of the natural numbers,
- $H \wr_X Sym(X)$, where H is finite and X is an infinite set,
- ...

Musings, open questions

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2. Compute the conjugacy growth series for other groups: surface, Artin, Coxeter ...

Musings, open questions

1. Can we find good bounds for the conjugacy growth of (some) relatively/acylindrically hyperbolic groups?
2. Compute the conjugacy growth series for other groups: surface, Artin, Coxeter ...
3. Are there groups with algebraic conjugacy growth series?

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4. How do the growth functions and growth series studied here behave when we change the set of generators?

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   Stoll: The rationality of the standard growth series depends on the generating set.

5. Are there groups, besides the abelian and virtually cyclic ones, with rational conjugacy growth series?

Thank you!