Topology of large random spaces

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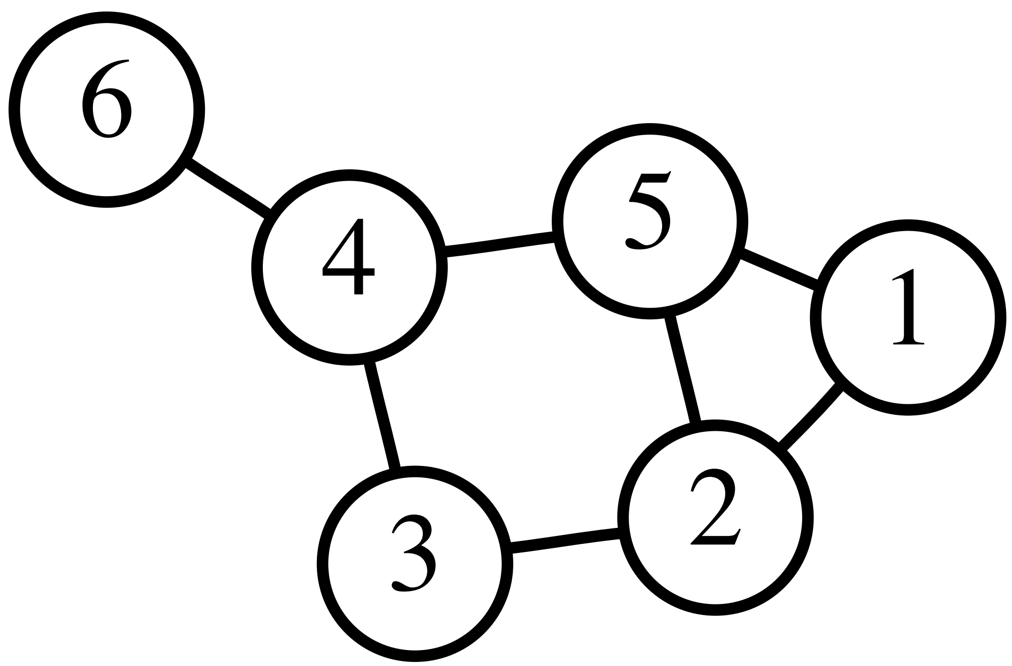
Informal definition:

a large random space is a topological space (typically, a simplicial complex or a manifold) which is the product of many independent random choices.

$$
\begin{tikzcd}
\text{Classical mechanics} \arrow[r, dotted] \arrow[d] & \text{Deterministic topology} \arrow[d] \\
\text{Statistical mechanics} \arrow[r, dotted] & \text{Stochastic topology}
\end{tikzcd}
$$

Complex networks as simplicial complexes



Complex networks are commonly represented by means of simple or directed **graphs** consisting of sets of **nodes** (representing **objects** under investigation) joined together in pairs by edges if the corresponding objects are in relationship of some kind.

It has been recently realised that in some practical applications representation of complex systems by graphs is inappropriate since crucial role play not only pairwise interactions but also interactions between more than two objects.

In such cases one may use **simplicial complexes** (or hyper-graphs) to represent the connections existing in the given system.

Example: market transactions are often characterised by participation of a buyer, a seller and a broker.

In my talk I will present a *mathematician's* view on large random spaces.

I will be mainly interested in geometric and topological properties of large random spaces and will describe several mathematical models which appeared in the mathematical literature.

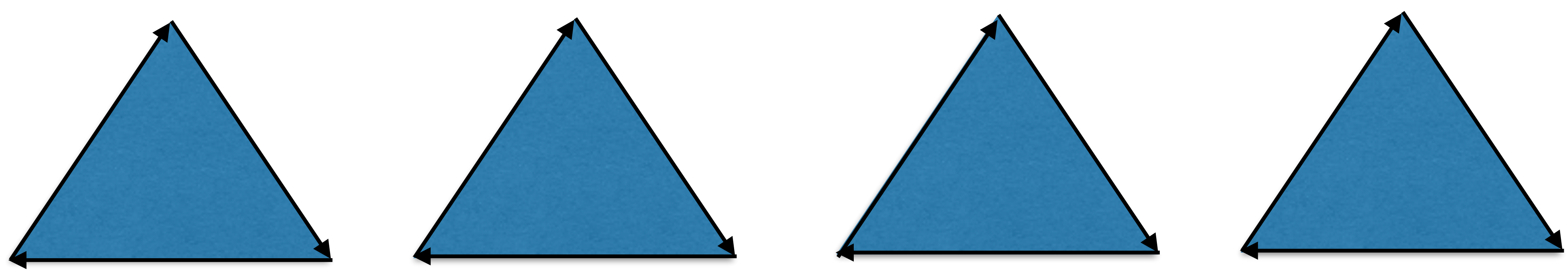
For a mathematician, large random spaces may also be attractive as a source of curious examples with rare combination of properties. In my talk I will discuss properties of random simplicial complexes in connection to the well-known **Whitehead** and **Eilenberg - Ganea Conjectures**.

Models of Stochastic Topology

1. Random surfaces.
2. Random 3-dimensional manifolds.
3. Random configuration spaces.
4. Random simplicial complexes - the Linial - Meshulam model.
5. Random aspherical 2-dimensional complexes and the Whitehead and Eilenberg - Ganea Conjectures.
6. Multi-parameter random simplicial complexes.
7. Conclusions

Random Surfaces

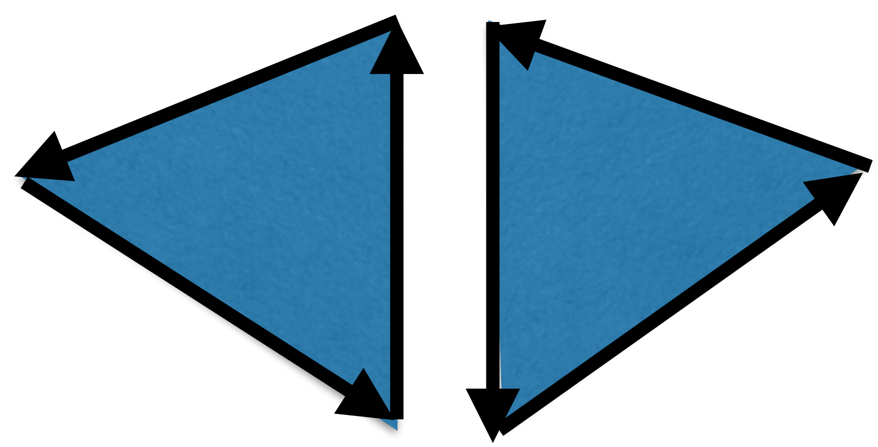
Pippenger and Schleich, 2006



Start with n oriented triangles, where n is even, n tends to infinity.

Pair their 3n edges at random.

Then glue these n triangles together using the edge pairing respecting the orientation.



The result is an orientable surface Σ.

How many ways to pair $3n$ edges?

$$\frac{3n!}{2^{3n/2}\cdot(3n/2)!}=(3n-1)!!=(3n-1)\cdot(3n-3)\cdot\ldots\cdot 1.$$

Each such surface Σ appears with probability

$$\left((3n-1)!!\right)^{-1}.$$

Let $h = h(\Sigma)$ denote the number of vertices of Σ. Then $\chi = h - 3n/2 + n = h - n/2$.

With probability

$$1 - \frac{5}{18n} + O\left(\frac{1}{n^2}\right)$$

the surface Σ is connected.

The genus of Σ equals

$$1 + \frac{n}{4} - \frac{h}{2}.$$

Pippenger and Schleich:

With probability tending to 1 the surface Σ is connected and its genus g has the expected value

$$E(g) = \frac{n}{4} - \frac{1}{2}\log n + O(1).$$

Pippenger and Schleich conjectured:

$$E(h) = \log(3n) + \gamma + o(1),$$

$$Var(h) = \log(3n) + \gamma - \frac{\pi^2}{6} + o(1)$$

where

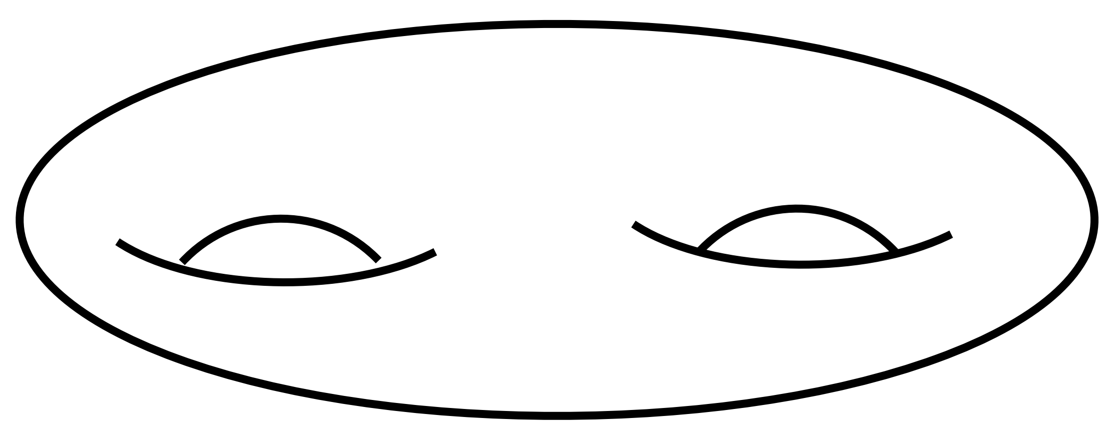
$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log n\right] = 0.57721...$$

is the Euler-Mascheroni constant.

Random 3-manifolds

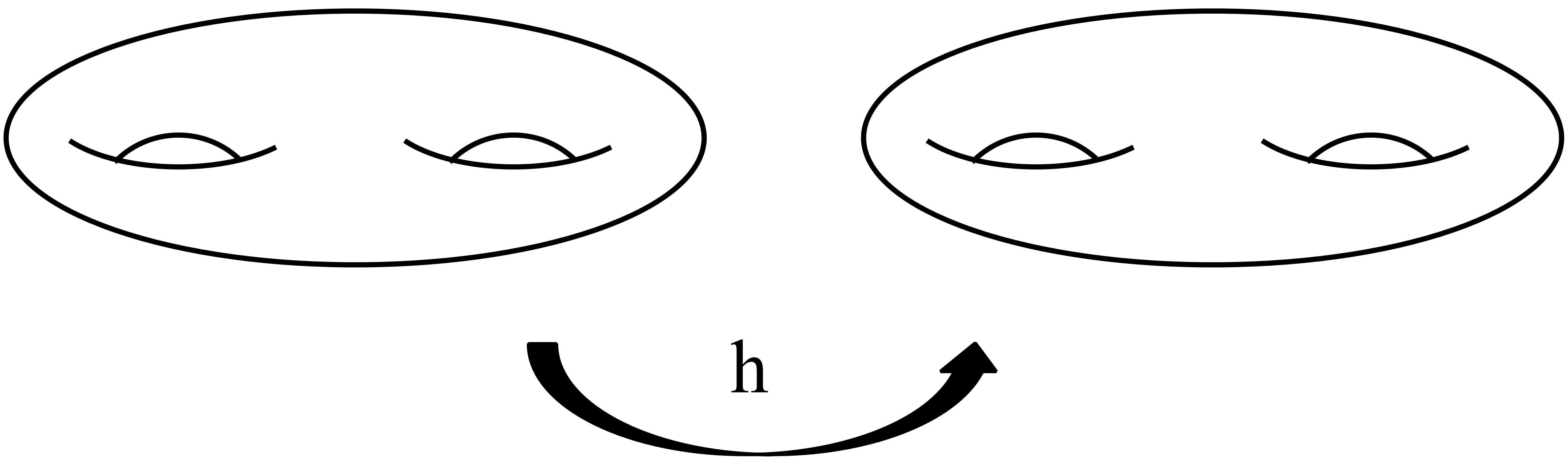
Dunfield, Thurston, 2006

Fix $g \geq 0$. Consider a genus g handlebody H_g and its boundary surface
$\Sigma_g = \partial H_g$.



A closed 3-manifold M_h can be obtained by gluing together two copies H_g and H'_g via a homeomorphism

$h: \partial H_g \to \partial H'_g$, $\quad M_h = H_g \cup_h H'_g$.



Let M_g denote the mapping class group of the surface Σ_g.

Fix a set of generators $S \subset M_g$, with $S = S^{-1}$.

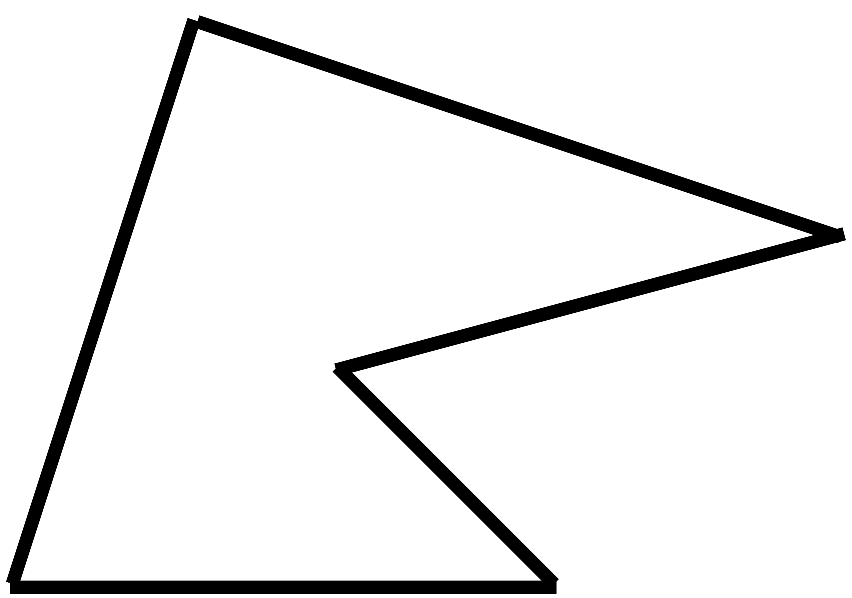
A random element $h \in M_g$ is defined as the result of a random walk in the generating set S, of length L.

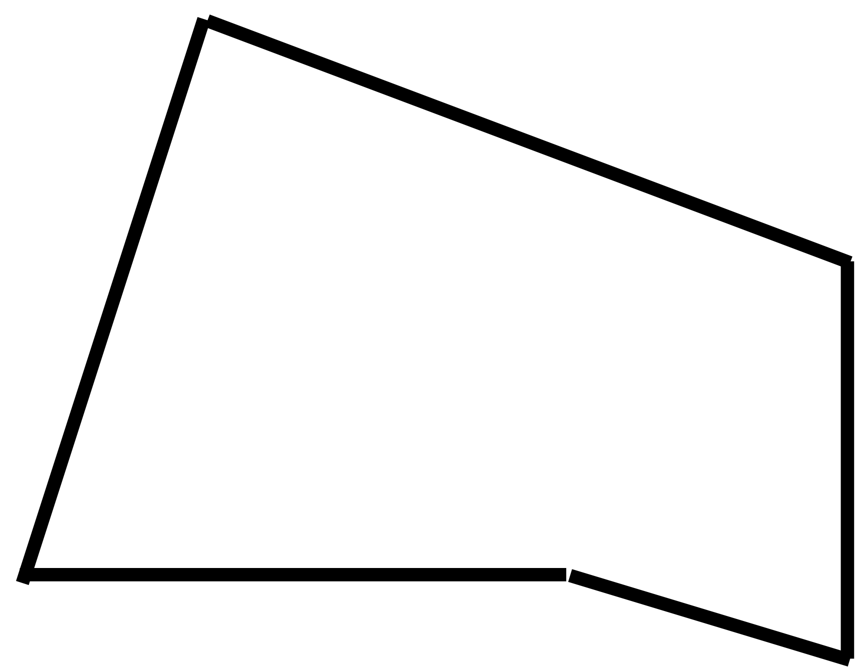
Properties of the 3-manifold M_h when h is random of length L where $L \to \infty$.

A random closed 3-manifold M_h
is obtained by glueing together two handlebodies
H_g and H'_g via a random homeomrphism
$h: \Sigma_g \to \Sigma_g$.

Random polygon spaces

Fix a vector $l = (l_1, \ldots, l_n)$, where $l_i > 0$.
Consider shapes of planar $n-$gons with sides of length $l_1, l_2, \ldots, l_n$.





Formally $\mathcal{M}_\ell$ is defined as follows

$$\mathcal{M}_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots S^1; \sum_{i=1}^{n} \ell_i u_i = 0\}/\mathrm{SO}(2).$$

If the length vector $\ell = (\ell_1, \dots, \ell_n)$ is generic then M_ℓ is a closed smooth manifold of dimension $n-3$.

Vector $\ell = (\ell_1, \dots, \ell_n)$ is *generic* if

$$\sum_{i=1}^{n} \epsilon_i \ell_i \neq 0, \quad \epsilon_i = \pm,$$

i.e. when there are no collinear configurations.

Case $n = 3$ and ℓ generic:

$\mathcal{M}_\ell$ can be either the two-point set S^0 or $\emptyset$.

Case $n = 4$ and ℓ generic:

$\mathcal{M}_\ell$ can be either S^1 or $S^1 \sqcup S^1$ or $\emptyset$.

Case $n = 5$ and ℓ generic:

$\mathcal{M}_\ell$ can be one of the following

$$\emptyset, \quad T^2 \sqcup T^2, \quad \Sigma_g$$

where Σ_g is a compact orientable surface of genus $g = 0, 1, 2, 3, 4$.

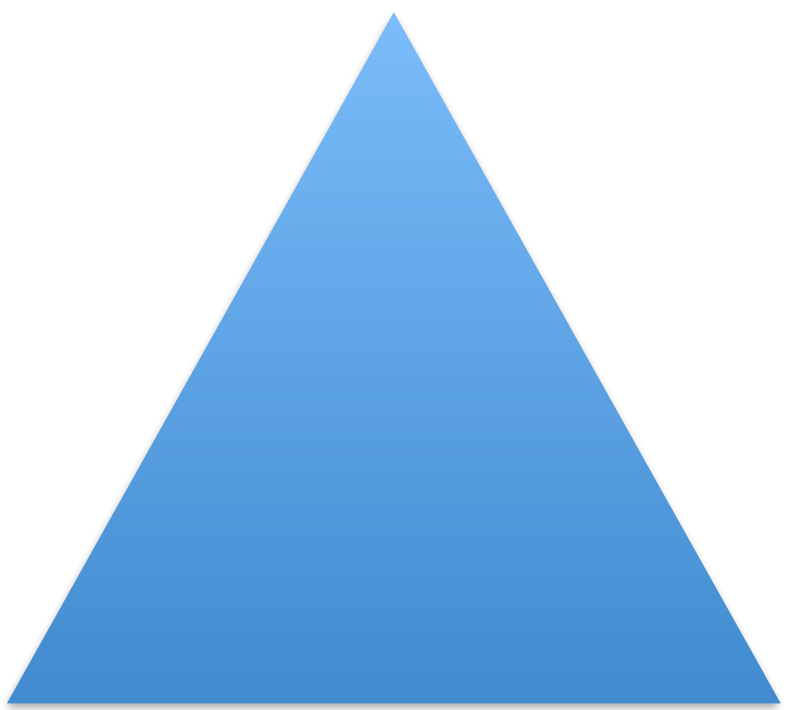
Classification of the manifolds $\mathcal{M}_\ell$ for small n was done by Kevin Walker, Misha Kapovich, John Millson and others.

How many distinct manifolds $\mathcal{M}_\ell$ we obtain for a fixed n by varying generic vector $\ell = (\ell_1, \ldots, \ell_n)$?

n	3	4	5	6	7	8	9	10
c_n	2	3	7	21	135	2,470	175,428	52,980,624

J-Cl. Hausmann, E.Rodriguez

What happens when n tends to infinity?



Given a probability measure on the unit simplex $\Delta \subset \mathbb{R}_+^n$, one may sample points $l \in \Delta$ at random, with respect to this measure. Then M_l becomes a random manifold.

Surprisingly, dependence of some topological properties of M_l on the measure dissappears in the limit $n \to \infty$.

Fix an integer p and consider the p-dimensional Betti number $b_p(\mathcal{M}_\ell)$ as a random variable on Δ,

$$b_p : \Delta \to \mathbb{R}.$$

Its expectation is given by

$$\mathbb{E}(b_p) = \int_\Delta b_p(\mathcal{M}_\ell) d\mu_n(\ell)$$

where $\mu_n(\ell)$ is the selected measure on Δ.

Theorem: [T. Kappeler, C. Dombry, C. Mazza, M.F.]:

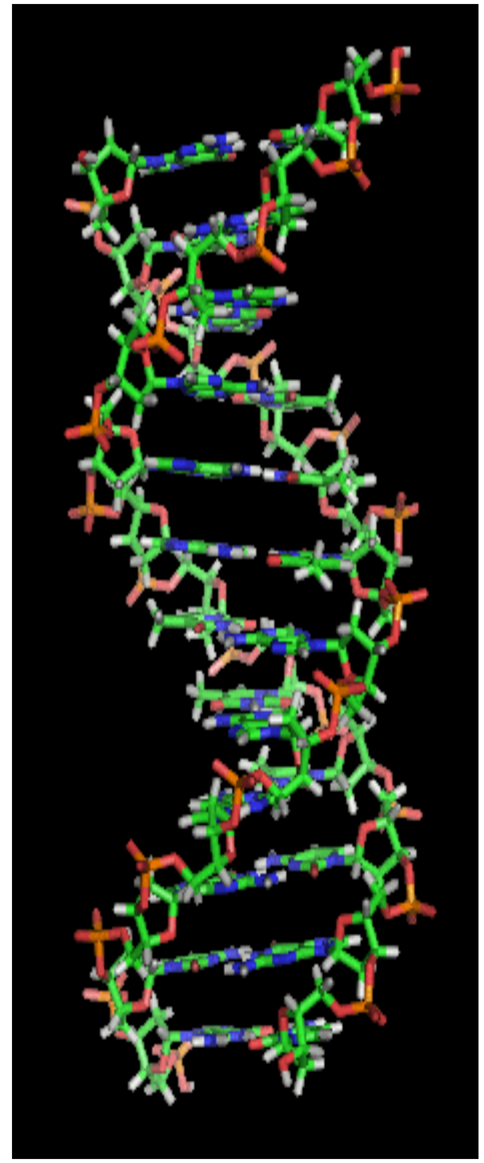
For a "vast class" of probability measures μ_n, the expectation $\mathbb{E}(b_p)$ equals

$$\binom{n-1}{p},$$

up to an exponentially small error $< e^{-cn}$, where $c > 0$ is a constant.

Note that

$$\binom{n-1}{p} = b_p(\mathcal{M}_{(1,1,,\ldots,1)}).$$



One may expect that similar phenomena occur for configuration spaces of more general linkages, including molecules like this.

Random simplicial complexes

In 2006 Linial and Meshulam initiated the topological study of high dimensional random simplicial complexes.

In their model, one starts with a complete graph on n vertices which are labelled by the integers

$$\{1, \ldots, n\}$$

and adds each triangle (ijk) at random, independently of the others, with probability $p \in [0, 1]$.

As the result we obtain a random 2-dimensional simplicial complex on n vertices.

Formally the probability space $Y_2(n,p)$ of the LM model consists of all subcomplexes

$$\Delta^{(1)} \subset Y \subset \Delta^{(2)}$$

and the probability function

$$P : Y_2(n,p) \to \mathbb{R}$$

is given by

$$P(Y) = p^{f_2(Y)}(1-p)^{\binom{n}{3}-f_2(Y)}.$$

We say that a property of a random simplicial complex

$$Y \in Y_2(n, p)$$

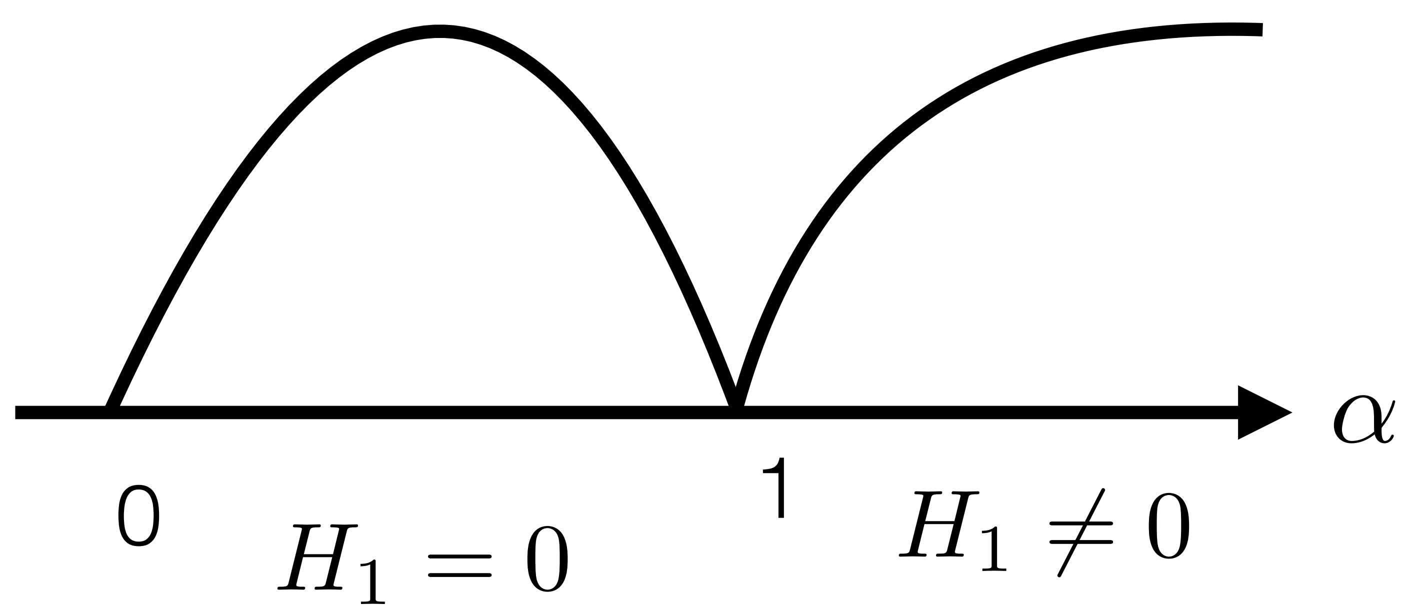
holds asymptotically almost surely (a.a.s) if the probability of its occurrence tends to 1 as $n \to \infty$.

Theorem (Linial and Meshulam, 2006):
Let ω be a sequence of real numbers tending to ∞.

If $p \geq \frac{2\log n + \omega}{n}$
then a random 2-complex $Y \in Y_2(n,p)$ satisfies
$H_1(Y, \mathbb{Z}_2) = 0$, a.a.s.

If $p \leq \frac{2\log n - \omega}{n}$
then a random 2-complex $Y \in Y_2(n,p)$ satisfies
$H_1(Y, \mathbb{Z}_2) \neq 0$, a.a.s

We may illustrate this theorem by the following picture.
We shall assume for simplicity that
$p = n^{-\alpha}$
where $\alpha \geq 0$.
Then $H_1 = 0$ for $\alpha \in [0, 1)$
and $H_1 \neq 0$ for $\alpha \in (1, \infty)$.



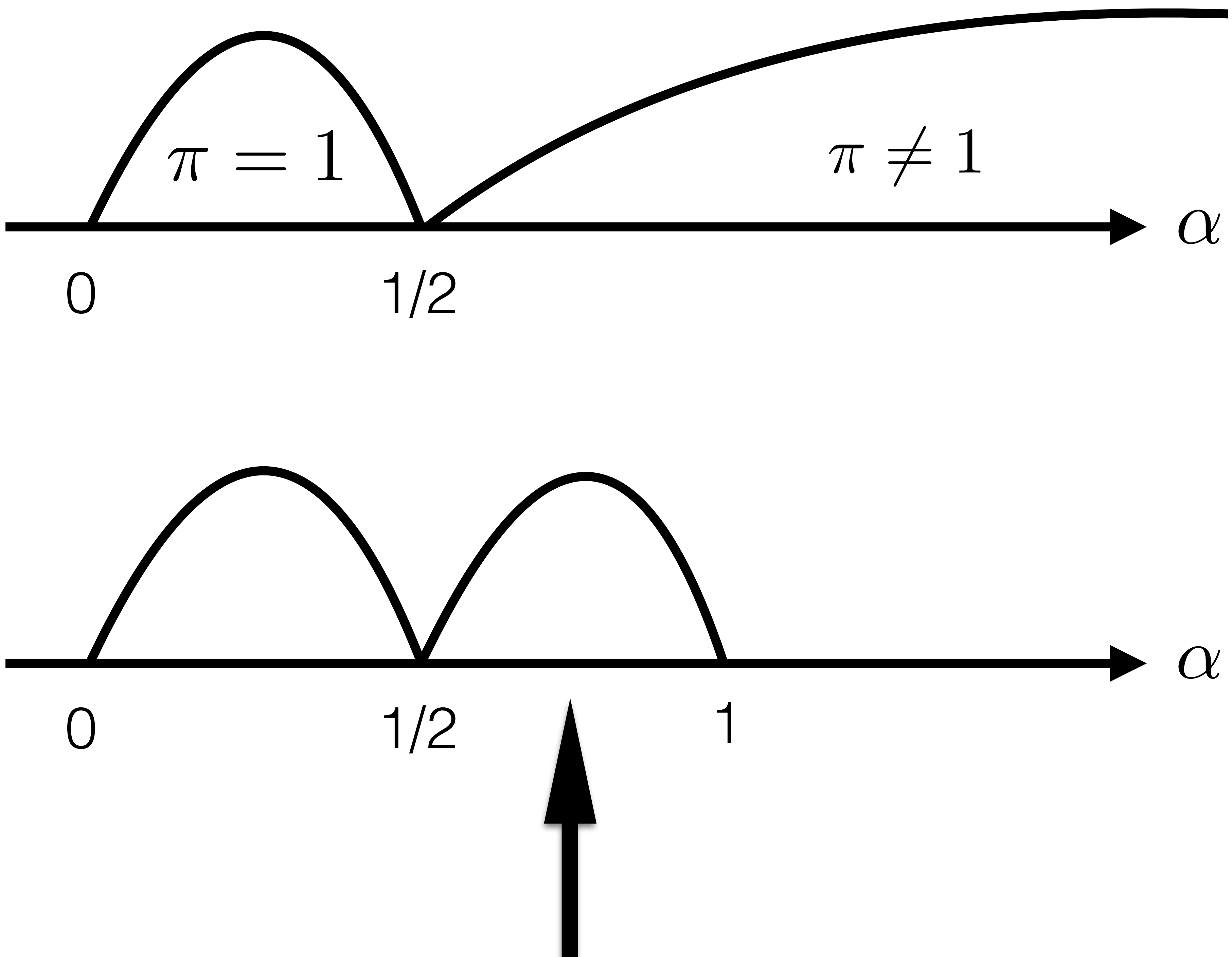
The fundamental group of a random 2-complex $Y \in Y_2(n, p)$ was studied in a significant paper of E. Babson, C. Hoffman, and M. Kahle (2011). They showed that:

$$\pi_1(Y) = 1 \text{ for } \alpha \in [0, 1/2), \text{ a.a.s.}$$

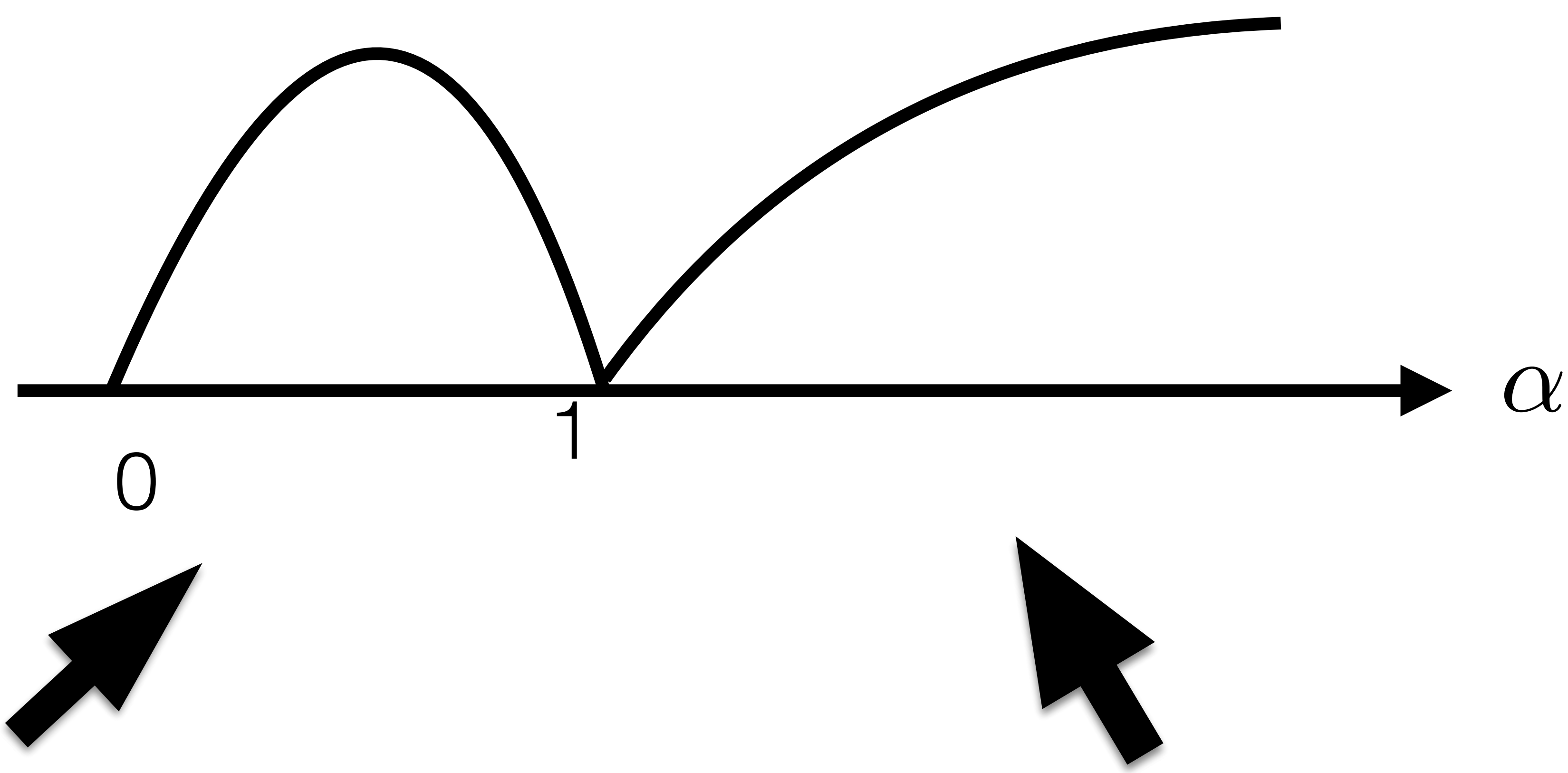
and

$$\pi_1(Y) \neq 1 \text{ for } \alpha \in (1/2, \infty), \text{ a.a.s.}$$

Moreover for $\alpha \in (1/2, \infty)$ the fundamental group is *hyperbolic* in the sense of Gromov.



$\pi_1(Y)$ is a nontrivial random perfect group.

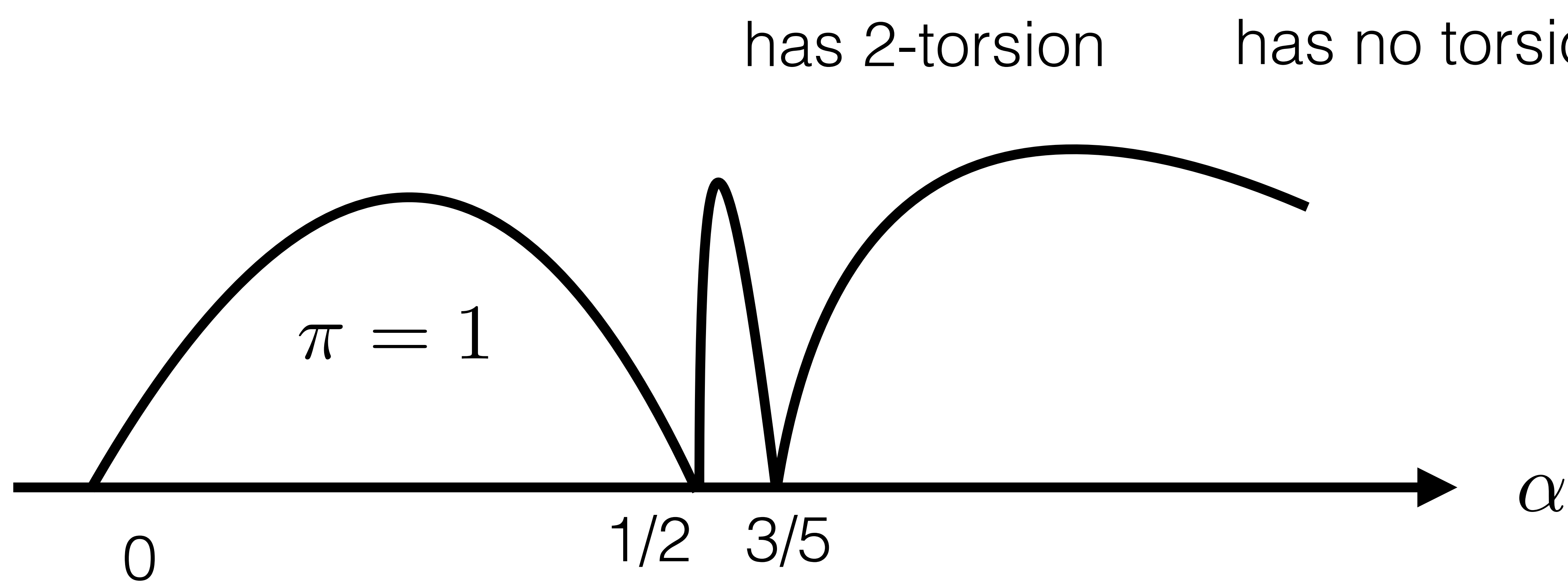


$\pi_1(Y)$ has property T.
(M. Kahle).

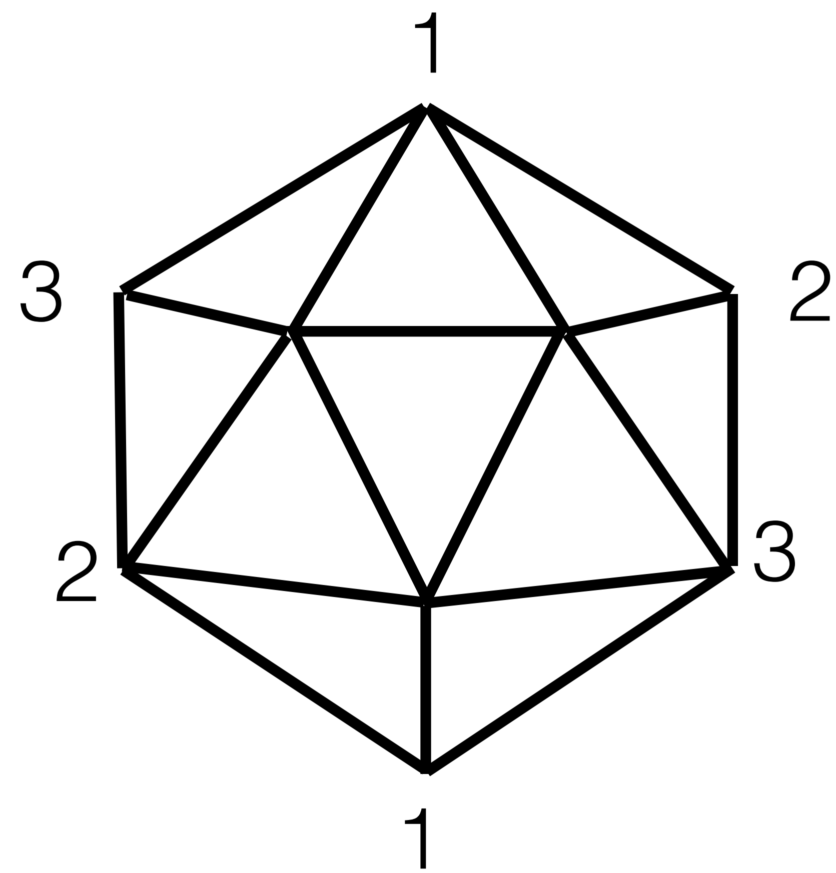
$\pi_1(Y)$ is free
(Cohen, Costa, Kappeler, MF).

Torsion in the fundamental group

Consider the fundamental group $\pi_1(Y)$ of a random simplicial complex where $Y \in Y_2(n,p)$, $p = n^{-\alpha}$.



A. Costa and MF.



Triangulation S of the real projective plane with 6 vertices and 10 faces
6/10=3/5=v/f

To prove the existence of 2-torsion we show that for

$$1/2 < \alpha < 3/5$$

the complex S embeds into a random 2-complex
$Y \in Y_2(n, p = n^{-\alpha})$
in an essential way, i.e. such that the induced map

$$\pi_1(S) = \mathbb{Z}_2 \to \pi_1(Y)$$

is an injection.

The proof uses the following tools:

1. Uniform hyperbolicity of random simplicial complexes;

2. Topological classification of minimal cycles;

3. Relative embedding theory.

Uniform hyperbolicity of random complexes

Let X be a simplicial complex.
Let $\gamma : S^1 \to X^{(1)}$ be a simplicial null-homotopic loop.
We denote by $|\gamma|$ the length of γ and by $A(\gamma)$ the area of γ, i.e. the number of triangles in a minimal spanning disc for γ.
The number

$$I(X) = \inf\{\tfrac{|\gamma|}{A(\gamma)}; \gamma : S^1 \to X,\ \gamma \sim 1\}$$

is the *isoperimetric constant of* X.

$\pi_1(X)$ is hyperbolic in the sense of Gromov if $I(X) > 0$.

Theorem:

Suppose that $p = n^{-\alpha}$ where $\alpha > 1/2$.
Then there exists a constant $c = c_\alpha$ such that with probability tending to 1 as $n \to \infty$ a random complex $Y \in Y_2(n, n^{-\alpha})$ has the following property:
any subcomplex $Y' \subset Y$ satisfies

$I(Y') \geq c_\alpha;$

in particular, the fundamental group $\pi_1(Y')$ is hyperbolic.

Cohomological and geometric dimension

Given a discrete group π, its geometric dimension is defined as the minimal dimension of an aspherical cell complex having π as its fundamental group.

The cohomological dimension $cd(\pi)$ is defined as the minimal length of a free resolution of $\mathbb{Z}$ over the group ring $\mathbb{Z}[\pi]$.

We know that

the equality

$$\mathrm{gdim}(\pi) = \mathrm{cd}(\pi)$$

takes place except possibly the case when

$\mathrm{gdim}(\pi) = 3$ and $\mathrm{cd}(\pi) = 2$.

The well-known *Eilenberg - Ganea Conjecture*

states that the case

$\mathrm{gdim}(\pi) = 3$ and $\mathrm{cd}(\pi) = 2$

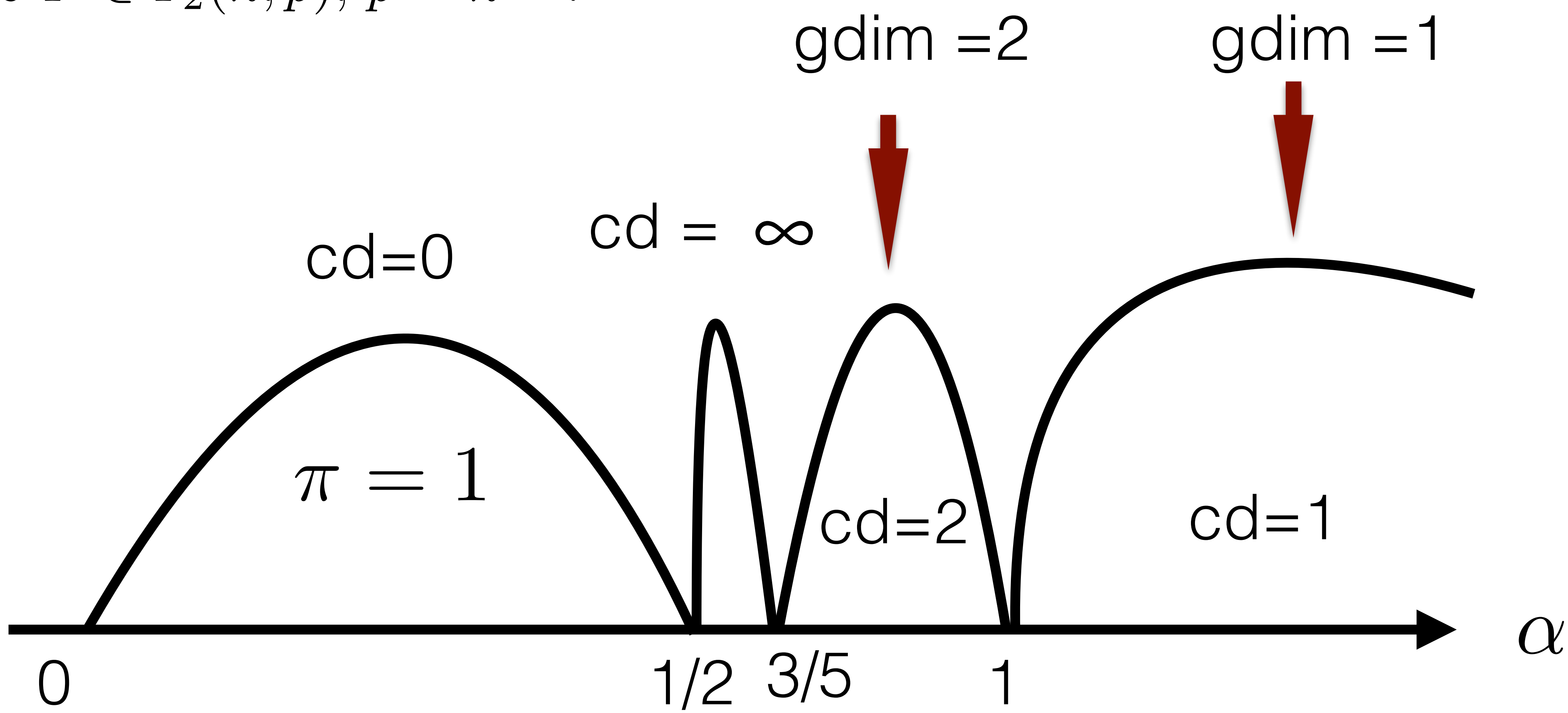
is impossible and therefore

$\mathrm{gdim}(\pi) = \mathrm{cd}(\pi)$.

for any group π.

Cohomological and geometric dimension of the fundamental groups of random complexes

Consider the fundamental group $\pi_1(Y)$ of a random simplicial complex where $Y \in Y_2(n,p)$, $p = n^{-\alpha}$.



A. Costa and MF.

Odd torsion

Theorem:

For any odd prime m, and for any
$\alpha \neq 1/2$
the fundamental group
of a random 2-complex $Y \in Y_2(n, p)$, with
$p = n^\alpha$,
has no m-torsion, a.a.s.

A. Costa, MF

The Whitehead Conjecture

Let X be a 2-dimensional finite simplicial complex.

X is called *aspherical* if $\pi_2(X) = 0$.

Equivalently, X is aspherical if the universal cover $\tilde{X}$ is contractible.

Examples of aspherical 2-complexes: Σ_g with $g > 0$;

N_g with $g > 1$.

Non-aspherical are S^2 and P^2 (the real projective plane).

In 1941, J.H.C. Whitehead suggested the following question:

Is every subcomplex of an aspherical 2-complex also aspherical?

This question is known as the Whitehead conjecture.

Can one test the Whitehead Conjecture probabilistically?

Tasks :

1. Produce aspherical 2-complexes randomly;
2. Estimate the probability that the Whitehead Conjecture is satisfied

Theorem : Let $Y \in Y(n,p)$ be a random 2-complex in the LM model, where

$p = n^{-\alpha}, \quad \alpha > \mathbf{1/2.}$

Then with probability tending to 1 as $n \to \infty$

Y has the following property:

a subcomplex $Y' \subset Y$ is aspherical if and only if

any subcomplex $S \subset Y'$ with at most $\mathbf{2}\varepsilon^{-1}$ faces is aspherical,

where $\varepsilon = \alpha - \mathbf{1/2.}$

A. Costa, MF

Corollary :

If $p = n^{-\alpha}$, where $\alpha > 1/2$, then a random 2-complex $Y \in Y(n, p)$ with probability tending to one as $n \to \infty$ has the following property:

any aspherical subcomplex $Y' \subset Y$ satisfies the Whitehead Conjecture, i.e. all subcomplexes $Y'' \subset Y'$ are also aspherical.

A. Costa, MF

More general random simplicial complexes

I will describe the approach we developed in a joint work with A. Costa.

A slightly different approach was suggested by K. Zuev, O. Eisenberg and D. Krioukov.

There was also an interesting recent preprint by C. Fowler.

Fix an integer $r > 0$.

We want to generate randomly r-dimensional subcomplexes of Δ_n - simplex with vertex set $\{1, 2, \ldots, n\}$.

Denote by Ω_n^r the set of all subcomplexes $Y \subset \Delta_n^{(r)}$.

To define the probability function we fix a sequence
$\mathfrak{p} = (p_0, p_1, \ldots, p_r), \quad p_i \in [0, 1]$.

For $Y \in \Omega_n^r$ we set

$$P(Y) = \prod p_i^{f_i(Y)} \cdot \prod q_i^{e_i(Y)}$$

where $q_i = 1 - p_i$.

Here $f_i(Y)$ denotes the number of i-dimensional faces contained in Y and

$e_i(Y)$ denotes the number of *external* i-dimensional faces of Y.

An external face is a simplex $\sigma \subset \Delta_n$ such that $\sigma \not\subset Y$ but $\partial\sigma \subset Y$.

One shows that

$$\sum_{Y \in \Omega_n^r} P(Y) = 1$$

i.e. we indeed have a probability measure.

Important Special Cases:

$\mathfrak{p} = (1, p)$ - ER random graphs,

$\mathfrak{p} = (1, 1, p)$ - LM random 2-complexes,

$\mathfrak{p} = (1, p, 1, \ldots, 1)$ - clique complexes of random graphs.

Gibbs Formalism

A more general class of probability measures can be constructed as follows. Consider an energy function

$H : \Omega_n^r \to \mathbb{R},$

$H(Y) = \sum_{i=0}^{r} \left[\beta_i f_i(Y) + \gamma_i e_i(Y)\right].$

Then

$P(Y) = Z^{-1} e^{H(Y)}$

is a probability measure on Ω_n^r, where

$Z = \sum_{Y \in \Omega_n^r} e^{H(Y)}$ is the partition function.

The multi-parameter probability measure is a special case

of Gibbs type measure,

it is obtained when the parameters β_i and γ_i are related

by the equation

$e_i^{\beta} + e_i^{\gamma} = 1.$

Then

$p_i = e^{\beta_i}$ and $q_i = e^{\gamma_i}$.

Betti numbers of random complexes

Surprisingly, homology of large random simplicial complexes is dominating in one specific dimension with all other Betti numbers of significantly smaller size.

We call this result *Domination Principle*.

Similar situation happens in the LM and in random clique complexes.

Domination Principle

Consider a random simplicial complex $Y \in \Omega_n^r$

with respect to the multi-parameter $\mathfrak{p} = (p_0, p_1, \ldots, p_r)$.

Let's assume that

$p_i = n^{-\alpha_i}$, where $\alpha_i \in \mathbb{R}_+$.

We obtain a multi-dimensional vector

$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_r) \in \mathbb{R}_+^{r+1}$ of exponents.

Consider the following linear functions:

$$\psi_k(\alpha) = \sum_{i=0}^{r} \binom{k}{i} \alpha_i, \qquad k = 0, \ldots, r,$$

where $\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathbb{R}^{r+1}$

Then

$$\psi_0(\alpha) \leq \psi_1(\alpha) \leq \psi_2(\alpha) \leq \ldots \leq \psi_r(\alpha).$$

Define the following convex domains (open sets) in $\mathbb{R}_+^{r+1}$:

$$\mathfrak{D}_k = \{\alpha \in \mathbb{R}_+^{r+1}\,;\, \psi_k(\alpha) < 1 < \psi_{k+1}(\alpha)\},$$

where $k = 0, 1, \ldots, r-1$.

One may also introduce the domains

$$\mathfrak{D}_{-1} = \{\alpha \in \mathbb{R}_+^{r+1}\, ;\, 1 < \psi_0(\alpha)\},$$

and

$$\mathfrak{D}_r = \{\alpha \in \mathbb{R}_+^{r+1}\, ;\, \psi_r(\alpha) < 1\}.$$

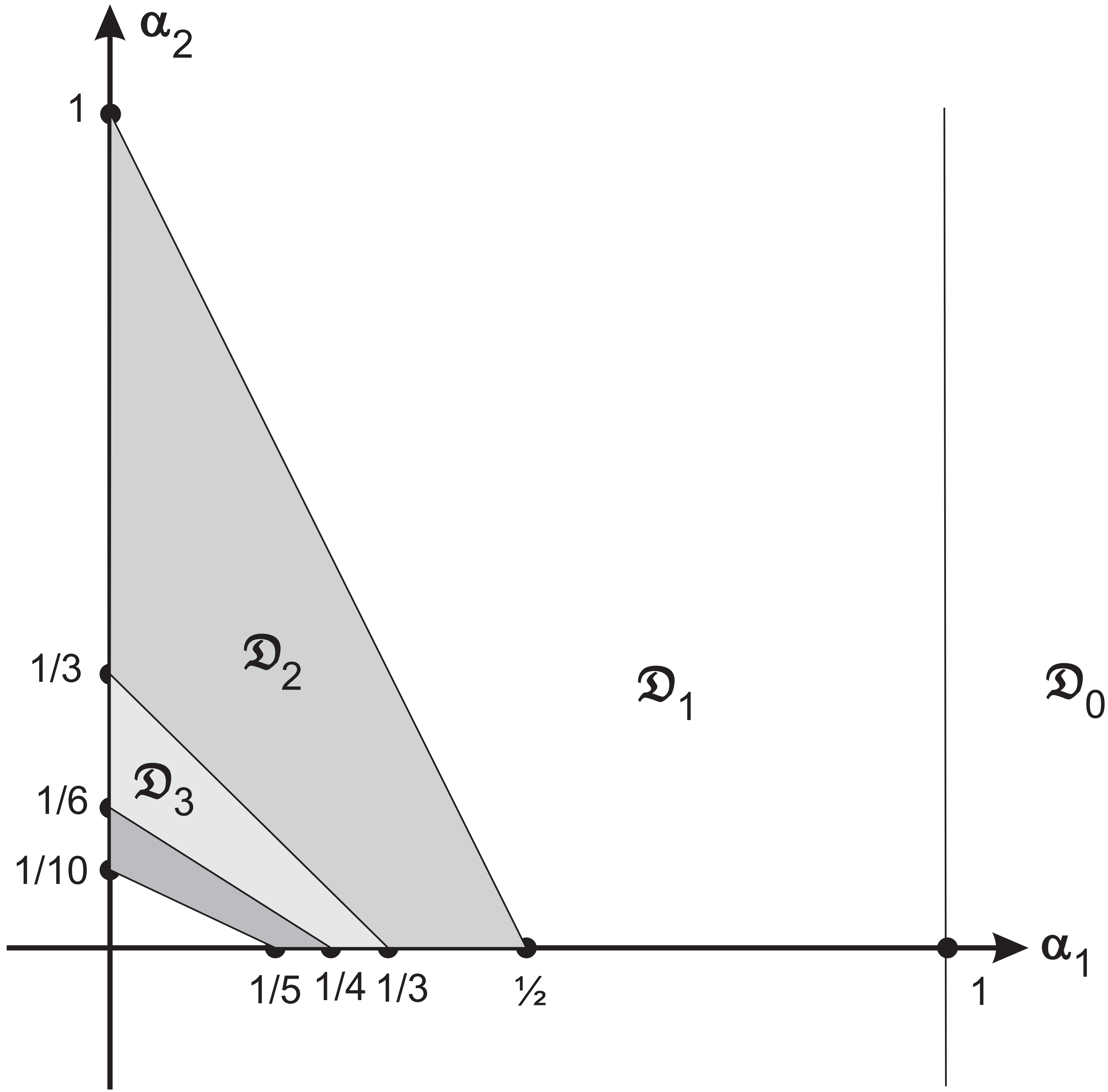
The domains $\mathfrak{D}_{-1}, \mathfrak{D}_0, \ldots, \mathfrak{D}_r$ are disjoint and their union is

$$\bigcup_{j=-1}^{r} \mathfrak{D}_j \; = \mathbb{R}_+^{r+1} - \bigcup_{i=0}^{r} H_i$$

where H_i denotes the hyperplane given by the linear equation $\psi_i(\alpha) = 1$.

Each hyperplane H_i correspond to *phase transitions in homology*.

If $\alpha \in \mathfrak{D}_{-1}$ then the random complex Y is $\emptyset$, a.a.s.



α_2
1
1/3
1/6
1/10
$\mathfrak{D}_2$
$\mathfrak{D}_3$
$\mathfrak{D}_1$
$\mathfrak{D}_0$
1/5
1/4
1/3
½
1
α_1

Next, we define a non-negative quantity $e(\alpha) = \min_k\{|1 - \psi_k(\alpha)|\}$.

Note that $e(\alpha) = \min\{1 - \psi_k(\alpha), \psi_{k+1}(\alpha) - 1\} > 0$ assuming that $\alpha \in \mathfrak{D}_k$.

Theorem:

For $\alpha \in \mathfrak{D}_k$ and n large enough one has

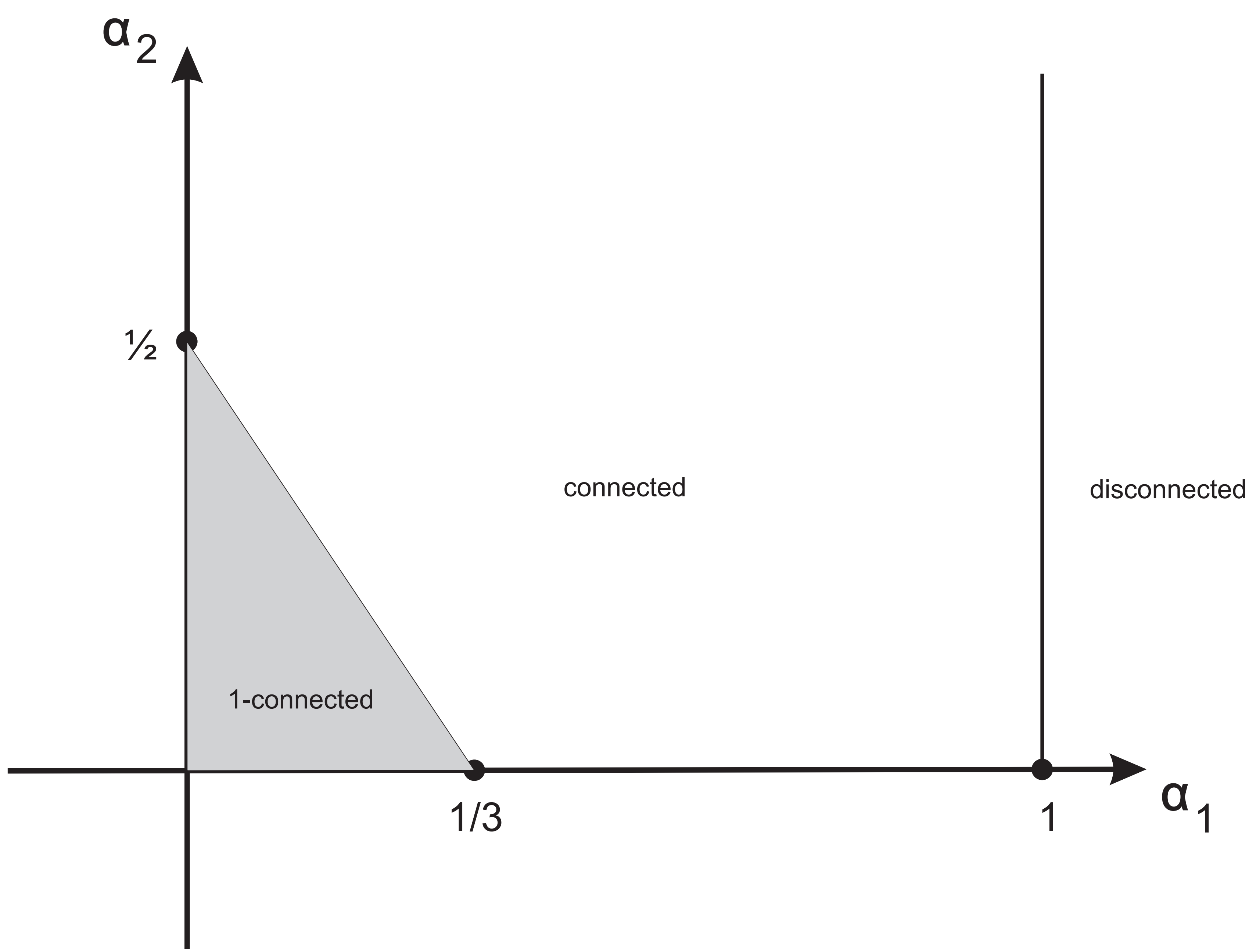
$$\mathbb{E}(b_k) \geq \frac{n^{e(\alpha)}}{(r+1)!} \cdot \mathbb{E}(b_j), \qquad j \neq k.$$

In other words, for $\alpha \in \mathfrak{D}_k$ the k-th Betti number significantly dominates all other Betti numbers.

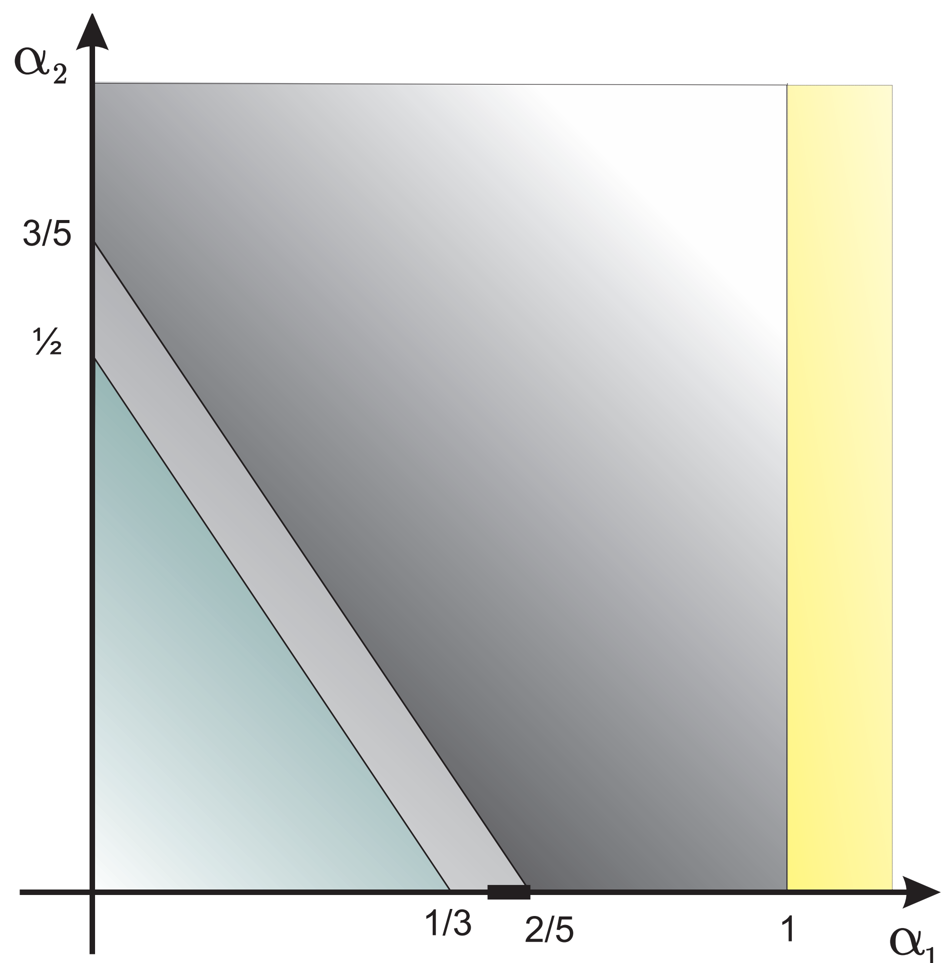
Besides,

$$\mathbb{E}(b_k) \sim \frac{1}{(k+1)!} \cdot n^{k+1-\sum_{i=0}^{k} \psi_i(\alpha)}.$$

Connectivity and simple connectivity



Areas on the plane α_1, α_2 corresponding to various properties of the fundamental group



(a) light green - the group is trivial;
(b) grey - the group has 2-torsion and is hyperbolic;
(c) shaded black (including the horizontal interval (11/30, 2/5) shown in bold) - the group is nontrivial, hyperbolic, its geometric dimension is \le 2;
(d) yellow - the group is trivial for any choice of the base point.

Conclusions

- Large random simplicial complexes tend to have their homology concentrated in one specific dimension.

- Fundamental groups of large random complexes may have cohomological dimension 1, 2, or infinity.

- Fundamental groups of large random spaces can be either free or have Kazhdan's property (T); and they may have only 2-torsion and no odd-torsion.

- Similar features appear in some other models producing random simplicial complexes.