Aalto University

Hyperbolic triangular buildings and periodic apartments

Geometry and Computation on Groups and Complexes Workshop

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Question

Gromov:

Does every one-ended hyperbolic group contain a subgroup which is isomorphic to the fundamental group of a closed surface?

Question

The answer is **yes** for

- Random groups (Calegari & Walker 2014)
- Groups acting on right-angled hyperbolic buildings (Futer & Thomas 2012)
- Groups acting on hyp. buildings with even-sided chambers (Vdovina 2005)
- Fundamental groups of hyperbolic 3-manifolds (Kahn & Marcovic 2012)
- Groups acting on negatively curved locally symmetric spaces, with some exceptions (Hamenstädt 2013)
- Right-angled Artin groups (Crisp, Sageev & Sapir 2008)

Question

We study surface subgroups of groups acting simply transitively on vertex sets of triangular hyperbolic buildings with the minimal generalized quadrangle as the link at each vertex.

We are especially interested in periodic apartments, invariant under an action of a surface group, since such an action implies the existence of a surface subgroup.

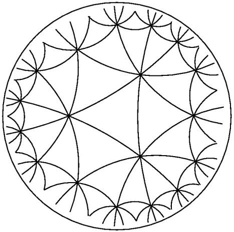
Existence of periodic apartments

- in Euclidean buildings, see Ballmann & Brin 1995
- in some hyperbolic buildings, see Vdovina 2005

Definitions

A spherical / euclidean(=affine) / hyperbolic Coxeter complex is a tiled sphere / euclidean space / hyperbolic plane where the tiles are closures of fundamental domains of finitely generated reflection groups.

We use the tessellation of the hyperbolic plane with triangles with all angles $\pi/4$.



Definitions

A *building* is a simplicial complex Δ which can be represented as the union of subcomplexes A, called *apartments*, satisfying the following axioms:

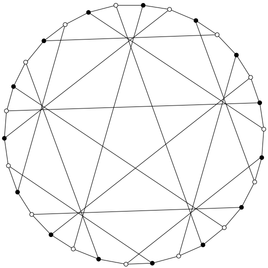
B0 Each apartment is a Coxeter group

B1 For any two simplices c, d in Δ there is an apartment A containing both of them

B2 If A and A' are two apartments containing simplices $c, d \in \Delta$, then there is an isomorphism $A \to A'$ fixing c and d point wise.

Triangular hyperbolic buildings

In 2010 K & Vdovina classified all torsion-free groups acting simply transitively on the vertices of hyperbolic triangular buildings of the smallest non-trivial thickness. Such buildings have the smallest generalized quadrangle GQ(2,2) as the link at each vertex.



Triangular hyperbolic buildings

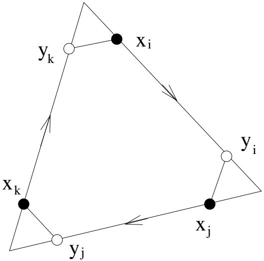
Theorem (Gaboriau & Paulin 2001)

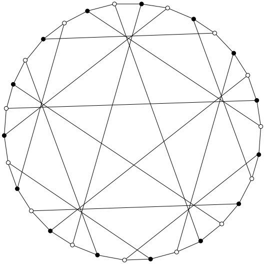
Let C_p be a polyhedron whose faces are p-gons and links are generalized m-gons with $mp > 2m + p$. Equip every face of C_p with the hyperbolic metric such that all sides of the polygons are geodesics and all angles are π/m. Then the universal covering of such a polyhedron is a hyperbolic building.

$\Rightarrow$ To construct hyperbolic buildings with cocompact group actions, it is sufficient to construct finite polyhedra with appropriate links.

Triangular hyperbolic buildings

Our buildings are constructed by finding one-vertex polyhedra consisting of 15 triangular faces with angles $\pi/4$ and having GQ(2,2) as the link, using the polygonal presentation method (Vdovina 2002).





Triangular hyperbolic buildings

Let P and Q be the sets of black and white vertices respectively in GQ(2,2). Then GQ(2,2) can be presented in the following way:

- "points" P are pairs (i, j), where $i, j = 1, ..., 6$, $i \neq j$
- "lines" Q are triples $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ of those pairs, where i_1, i_2, i_3, j_1, j_2 and j_3 are all different.

(Tits & Weiss 2002)

We denote the elements of P by x_i, $x_1 = (1, 2)$, $x_2 = (1, 3)$,..., $x_{15} = (5, 6)$ and the elements of Q by y_i, $i = 1, 2, \ldots, 15$.

$$
\begin{array}{ccccccc}
(12) & (34) & (56) & \Rightarrow & x_1 & x_{10} & x_{15} \\
(12) & (35) & (46) & & x_1 & x_{11} & x_{14} \\
(12) & (36) & (45) & & x_1 & x_{12} & x_{13} \\
(13) & (24) & (56) & & x_2 & x_7 & x_{15} \\
(13) & (25) & (46) & & x_2 & x_8 & x_{14} \\
(13) & (26) & (45) & & x_2 & x_9 & x_{13} \\
(14) & (23) & (56) & & x_3 & x_6 & x_{15} \\
(14) & (25) & (36) & & x_3 & x_8 & x_{12} \\
(14) & (26) & (35) & & x_3 & x_9 & x_{11} \\
(15) & (23) & (46) & & x_4 & x_6 & x_{14} \\
(15) & (24) & (36) & & x_4 & x_7 & x_{12} \\
(15) & (26) & (34) & & x_4 & x_9 & x_{10} \\
(16) & (23) & (45) & & x_5 & x_6 & x_{13} \\
(16) & (24) & (35) & & x_5 & x_7 & x_{11} \\
(16) & (25) & (34) & & x_5 & x_8 & x_{10}
\end{array}
$$

Triagonal presentations

Label the rows by $y_1, \ldots, y_{15}$ in such a way that the result is an incidence tableau of GQ(2,2) arising from 15 triangles.

Example:

$$
\begin{array}{llll}
y_1: & x_1 & x_{10} & x_{15}\\
y_2: & x_1 & x_{11} & x_{14}\\
y_{10}: & x_1 & x_{12} & x_{13}\\
y_3: & x_2 & x_7 & x_{15}\\
y_9: & x_2 & x_8 & x_{14}\\
y_{15}: & x_2 & x_9 & x_{13}\\
y_{14}: & x_3 & x_6 & x_{15}\\
y_4: & x_3 & x_8 & x_{12}\\
y_{13}: & x_3 & x_9 & x_{11}\\
y_6: & x_4 & x_6 & x_{14}\\
y_7: & x_4 & x_7 & x_{12}\\
y_{11}: & x_4 & x_9 & x_{10}\\
y_8: & x_5 & x_6 & x_{13}\\
y_{12}: & x_5 & x_7 & x_{11}\\
y_5: & x_5 & x_8 & x_{10}
\end{array}
$$

$$
\begin{array}{llll}
y_1 : & \mathbf{x_1} & \mathbf{x_{10}} & x_{15} \\
y_2 : & x_1 & x_{11} & x_{14} \\
y_{10} : & \mathbf{x_1} & x_{12} & x_{13} \\
y_3 : & x_2 & x_7 & x_{15} \\
y_9 : & x_2 & x_8 & x_{14} \\
y_{15} : & x_2 & x_9 & x_{13} \\
y_{14} : & x_3 & x_6 & x_{15} \\
y_4 : & x_3 & x_8 & x_{12} \\
y_{13} : & x_3 & x_9 & x_{11} \\
y_6 : & x_4 & x_6 & x_{14} \\
y_7 : & x_4 & x_7 & x_{12} \\
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y_{12} : & x_5 & x_7 & x_{11} \\
y_5 : & x_5 & x_8 & x_{10}
\end{array}
$$

y_1 :	x_1	x_{10}	x_{15}
y_2 :	x_1	x_{11}	x_{14}
y_{10} :	x_1	x_{12}	x_{13}
y_3 :	x_2	x_7	x_{15}
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y_8 :	x_5	x_6	x_{13}
y_{12} :	x_5	x_7	x_{11}
y_5 :	x_5	x_8	x_{10}

y_1 :	$\mathbf{x_1}$	$\mathbf{x_{10}}$	$\mathbf{x_{15}}$
y_2 :	$\mathbf{x_1}$	$\mathbf{x_{11}}$	x_{14}
y_{10} :	$\mathbf{x_1}$	x_{12}	x_{13}
y_3 :	x_2	x_7	x_{15}
y_9 :	$\mathbf{x_2}$	x_8	x_{14}
y_{15} :	$\mathbf{x_2}$	x_9	x_{13}
y_{14} :	x_3	x_6	x_{15}
y_4 :	x_3	x_8	x_{12}
y_{13} :	x_3	x_9	x_{11}
y_6 :	x_4	x_6	x_{14}
y_7 :	x_4	x_7	x_{12}
y_{11} :	x_4	$\mathbf{x_9}$	x_{10}
y_8 :	x_5	x_6	x_{13}
y_{12} :	x_5	x_7	x_{11}
y_5 :	x_5	x_8	x_{10}

y_1 :	x_1	x_{10}	x_{15}
y_2 :	x_1	x_{11}	x_{14}
y_{10} :	x_1	x_{12}	x_{13}
y_3 :	x_2	x_7	x_{15}
y_9 :	x_2	x_8	x_{14}
y_{15} :	x_2	x_9	x_{13}
y_{14} :	x_3	x_6	x_{15}
y_4 :	x_3	x_8	x_{12}
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y_6 :	x_4	x_6	x_{14}
y_7 :	x_4	x_7	x_{12}
y_{11} :	x_4	x_9	x_{10}
y_8 :	x_5	x_6	x_{13}
y_{12} :	x_5	x_7	x_{11}
y_5 :	x_5	x_8	x_{10}

Triangular hyperbolic buildings

The group with 15 generators $x_1, x_2, \ldots, x_{15}$ and the 15 words from the boundaries of the triangles as relations, acts on the building cocompactly and simply transitively.

There are 23 non-isomorphic groups without torsion.

Triangular hyperbolic buildings

T_1	T_3	T_9	T_{21}
(x_1, x_1, x_{10})	(x_1, x_1, x_{10})	(x_1, x_1, x_{10})	(x_1, x_5, x_2)
(x_1, x_{15}, x_2)	(x_1, x_{15}, x_2)	(x_1, x_{15}, x_2)	(x_4, x_{13}, x_{11})
(x_2, x_{11}, x_9)	(x_2, x_{11}, x_3)	(x_2, x_{11}, x_4)	(x_1, x_6, x_4)
(x_2, x_{14}, x_3)	(x_2, x_{14}, x_5)	(x_2, x_{14}, x_6)	(x_5, x_9, x_{10})
(x_3, x_7, x_4)	(x_3, x_7, x_4)	(x_3, x_5, x_9)	(x_1, x_3, x_{13})
(x_3, x_{15}, x_{13})	(x_3, x_{15}, x_8)	(x_3, x_8, x_7)	(x_5, x_{13}, x_9)
(x_4, x_8, x_6)	(x_4, x_8, x_9)	(x_3, x_{10}, x_{13})	(x_2, x_7, x_{10})
(x_4, x_{12}, x_{11})	(x_4, x_{12}, x_{12})	(x_4, x_8, x_5)	(x_6, x_9, x_8)
(x_5, x_5, x_8)	(x_5, x_9, x_6)	(x_4, x_{14}, x_{14})	(x_2, x_{12}, x_{15})
(x_5, x_{10}, x_{12})	(x_5, x_{13}, x_{13})	(x_5, x_{10}, x_{12})	(x_6, x_{11}, x_{10})
(x_6, x_6, x_{14})	(x_6, x_8, x_{11})	(x_6, x_7, x_{12})	(x_3, x_{11}, x_{14})
(x_7, x_7, x_{12})	(x_6, x_{10}, x_{13})	(x_6, x_{15}, x_9)	(x_7, x_8, x_{15})
(x_8, x_{13}, x_9)	(x_7, x_9, x_{14})	(x_7, x_8, x_{11})	(x_3, x_{14}, x_8)
(x_9, x_{14}, x_{15})	(x_7, x_{10}, x_{12})	(x_9, x_{15}, x_{13})	(x_7, x_{14}, x_{12})
(x_{10}, x_{13}, x_{11})	(x_{11}, x_{15}, x_{14})	(x_{11}, x_{12}, x_{13})	(x_4, x_{12}, x_{15})

Table : Presentations T_1, T_3, T_9 and T_{21} .

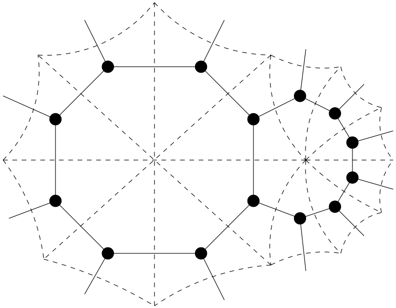
Triangular hyperbolic buildings

Theorem:

There are hyperbolic triangular buildings admitting simply-transitive torsion free action and having the smallest generalised quadrangle as the link at each vertex both with and without apartments invariant under genus 2 orientable surface group action.

Dual graphs

Assume that there is a periodic plane, and consider the dual graph. It is 3-valent, bipartite and has cycles of lenght 8.



$\Rightarrow$ # edges = 4 # octagons,
vertices = 8/3 # octagons

$\Rightarrow$ # octagons = 6g-6

If genus is two, the dualgraph has 16 vertices, 24 edges and 6 octagonal faces and thus is glued together from 16 triangles.

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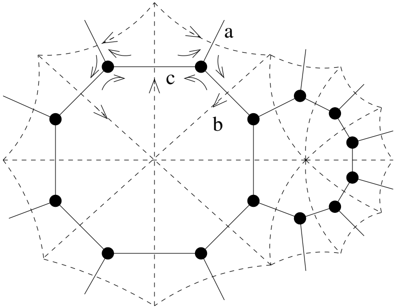

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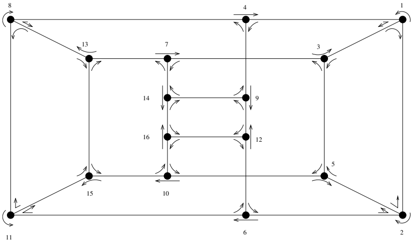
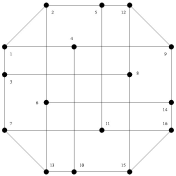
Dual graphs

These graphs are generated with `nauty` package (McKay & Piperno 2013) and a cycle search.

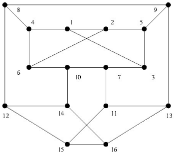
We get 12 candidates for possible dual graphs. They all have several possible orientations for the triangles that would give the six cycles of lenght 8.



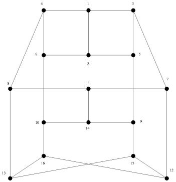
Dual graphs



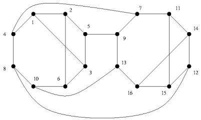
(a) G^0_{4060}



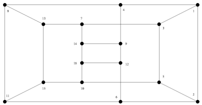
(b) G^0_{3538}



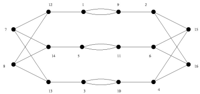
(c) G^0_{3621}



(d) G^0_{3345}

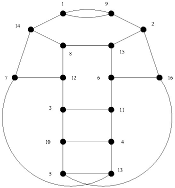


(e) G^0_{4002}

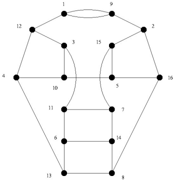


(f) G^3_{112}

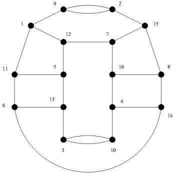
Dual graphs



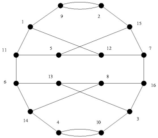
(a) G^1_{61}



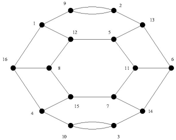
(b) G^1_{84}



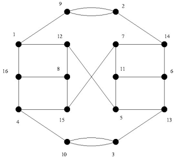
(c) G^2_{20}



(d) G^2_{25}



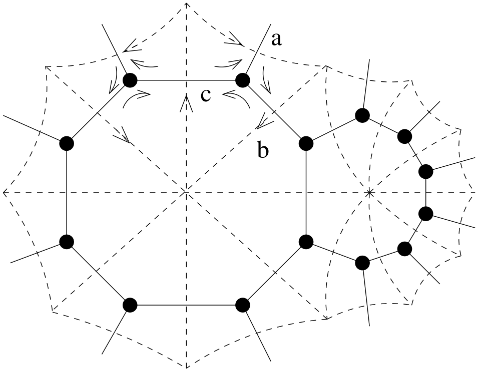
(e) G^2_{78}



(f) G^2_{84}

Dual graphs

In order to a surface to exists, the edges of the dual graph must be colourable by the triangles in the group.



Periodic apartments

G^0_{3345} is colourable with T_1 and T_2,

G^2_{78} and G^2_{85} are colourable with T_{18},

G^3_{112} is colourable with T_1, T_7 and T_9.

$\Rightarrow$ There are periodic apartments in the buildings that have T_1, T_2, T_7, T_9 or T_{18} acting on them.

The other 18 buildings do not have periodic apartments of genus 2.

Periodic apartments

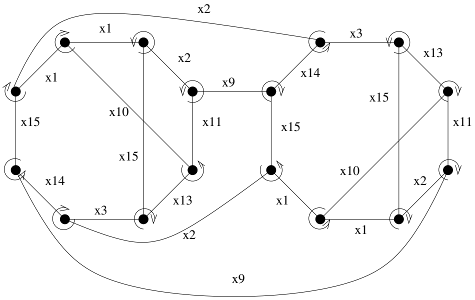
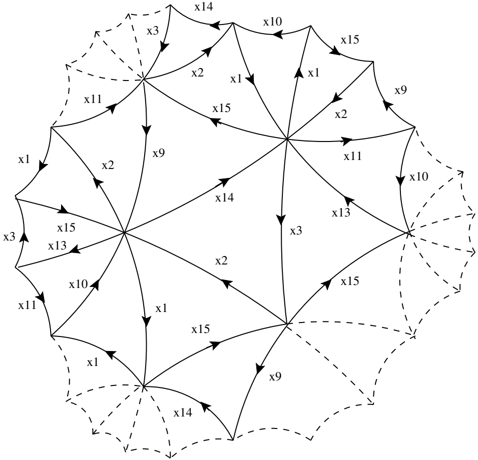


Figure : Graph G^0_{3345} coloured with the triangles from the group T_1

Periodic apartments



Periodic apartments

For genus 3, the 3-valent, bipartite dual graph has

- 32 vertices, 48 edges
- 12 cycles of lenght 8

Even without multiple edges there is $19 \cdot 10^{12}$ graphs to be checked for cycles (`nauty`, McKay& Piperno 2013).

$\Rightarrow$ Other ideas must be used, like boundary word graphs or choosing 8-cycles from the link to create the subgroup.

Some references

[1] R. Kangaslampi, and A .Vdovina, *Hyperbolic triangular buildings without periodic planes of genus two*, preprint 2015.

[2] L. Carbone, R. Kangaslampi, and A. Vdovina, *Groups acting simply transitively on vertex sets of hyperbolic triangular buildings*, LMS Journal of Computation and Mathematics 2012, 101–112.

[3] R. Kangaslampi and A. Vdovina, *Cocompact actions on hyperbolic buildings*, Internat. J. Algebra Comput. 20 (2010), no. 4, 591–603.