Lecture 1

An excellent accompanying book for the course is...

Kevin Houston: *How to think like a mathematician*, Cambridge

In this lecture we will introduce **a little bit of logic***. We will talk about statements and their combinations and truth tables.*

Mathematics in a nutshell: Thinking about problems, formulating mathematical statements, proving them!

Examples of mathematical statements:

- There are infinitely many prime numbers.
- $\sqrt{2}$ is not a rational number.

Mathematical statements are either true or false – but not both.

A false statement is still a perfectly fine statement: "17 is an even number." is still a perfectly fine mathematical statement even though it is false.

Two golden rules:

- It is important that all mathematical arguments are presented precisely and correctly.
- Mathematicians use formulas and symbols. But a too extensive use of abbreviations and symbols can make it difficult to understand their statements. Therefore, we should use them sparingly.

Further examples of statements and non-statements:

- If x, y, z are real numbers and $x < y$ and $y < z$, then $x < z$. *true statement*
- The sum of two odd numbers is odd. *false statement*
- If z_1, z_2 are complex numbers, then we have $z_1 \leq z_2$ or $z_2 \leq z_1$. *This is a nonsensical sentence, since complex numbers cannot be compared.*
- $x > 1$. *This is not a statement per se, but it becomes a statement whenever x is replaced by a real number. We call this an "OPEN STATEMENT".*

- Greece is a wonderful country! *This is not a mathematical statement.*

Combined statements and truth tables:

Statements can be combined with and, or, not, ... to form new statements. Whether these new statements are true or false can be derived from the information whether the statements A, B themselves are true or false. A useful tool to decide the outcome are **truth tables**. Mathematicians do not use them often, but they are helpful to learn logic.

Example: Durham has a cathedral **and** Oxford is a French city.

The first statement is true whilst the second is false. Combined with "and", both statements have to be true for the combined statement to be true. Therefore the combined statement is false.

Truth table for **and**:

A	B	A and B
False	False	False
False	True	False
True	False	False
True	True	True

Other **connectives**:

- Truth table for **not** (negation of a statement):

A	not A
False	True
True	False

- Truth table for **or**:

A	B	A or B
False	False	False
False	True	True
True	False	True
True	True	True

Finally, we discuss the **implies** connective: A true statement can lead to another true statement. Therefore, the following combined statement is true:

"$\pi > 3$ **implies** $-\pi < -3$".

A false statement can lead to any statement: Here a false statement leads to another false statement by adding 1 on both sides. The combined statement is true:

"$\pi < 3$ **implies** $\pi + 1 < 4$".

Here a false statement leads to a true statement by multiplying both sides with 0. The combined statement is true:

"$\pi = 3$ **implies** $0 = 0$".

But a true statement can never lead to a false one. We use the notation "$A \Rightarrow B$" for "A implies B" or, what is the same, for "If A then B." The truth table for this connective is

A	B	A $\Rightarrow$ B
False	False	True
False	True	True
True	False	False
True	True	True

Be aware: The combined statement "$\pi > 3$ implies 5 is a prime" seems to be false, since there is no natural connection between both statements "$\pi > 3$" and "5 is a prime" but, formally, this combined statement is a true statement.

Having created new combined statements we can define what we mean by "equivalent statements".

Definition. *Two combined statements are* **equivalent** *if their truth tables coincide.*

Example of equivalent statements: "$A \Rightarrow B$" is equivalent to "(not A) or B". We can verify this with the help of a truth table:

A	B	not A	(not A) or B	A $\Rightarrow$ B
False	False	True	True	True
False	True	True	True	True
True	False	False	False	False
True	True	False	True	True

Rules that can be checked by truth tables:

(a) "A and B" is equivalent to "B and A".

(b) "A and (B or C)" is equivalent to "(A and B) or (A and C)".

(c) "A or (B and C)" is equivalent to "(A or B) and (A or C)".

(d) "not (not A)" is equivalent to "A"

For friends of formal rules: (a) is the **law of commutativity** for "and" and (b) and (c) are **laws of distributivity** for the connectives "and" and "or".

Finally, we like to present a very useful fact.

Theorem (De Morgan's Rule)**.** *We have the following equivalences:*

(a) "(not A) and (not B)" is equivalent to "not (A or B)".

(b) "(not A) or (not B)" is equivalent to "not (A and B)".