Lecture 3

In the last lecture we introduced basics about sets and explained how to prove that two sets are equal and that a set is a subset of another set. In this lecture we will discuss how to create and to structure a mathematical text. In short, this lecture is concerned with the **writing of a mathematical text**. *Our examples will be taken from Euclidean Geometry, which is a good playground to learn good reasoning.*

Meaning of Definition, Theorem, Proof, ...

A mathematical text is structured by small "nuggets of information", to use a phrase by K. Houston. The most common structure elements are the following:

- **Definition:** Symbols and words for new mathematical objects are introduced in a definition.
- **Theorem:** A very important true statement (at least the author thinks so...)
- **Proposition:** A less important but nonetheless interesting true statement
- **Lemma:** a true statement used in proving other true statements
- **Corollary:** a true statement that is a consequence from a theorem or proposition
- **Proof:** the explanation of why a statement is true (proofs are either correct or wrong; there is no ambiguity)
- **Conjecture:** a statement believed to be true, but for which there is currently no proof.

Examples:

Definition. *A* prime number *is a natural number* $p > 1$ *whose only divisors are* 1 *and* p *itself.*

Theorem. *(Fermat's Last Theorem) There are no positive integer solutions for* a, b *and* c *to* $a^n + b^n = c^n$ *for* $n > 2$.

Corollary. *There are no positive rational solutions for* x, y *and* z *to* $x^n + y^n = z^n$ *for* $n > 2$.

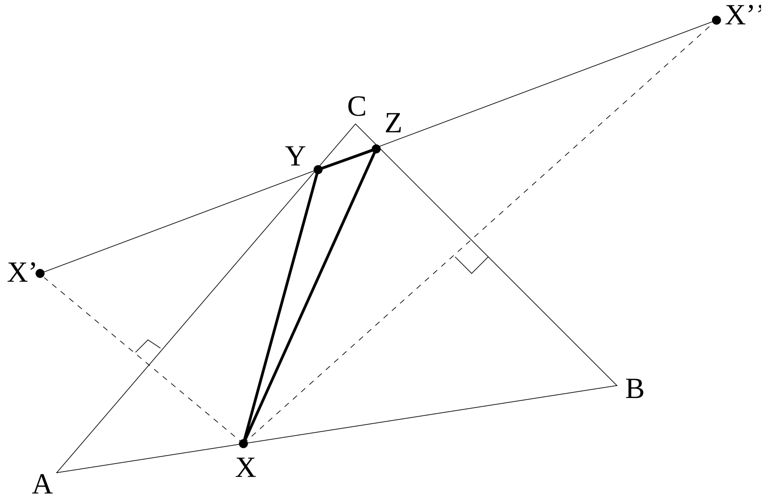
Conjecture. *(Goldbach Conjecture) Every even natural number* $n \geq 4$ *is the sum of two primes.*

We consider the following **geometric problem**:

Let ΔABC be an acute triangle and X be a fixed point on side $\overline{AB}$. We want to find a triangle ΔXYZ with endpoints on the three sides of ΔABC with minimal perimeter $\mathrm{perim}(\Delta XYZ)$*.*

We also consider the following **proposed solution** (see illustration):

Reflect X at the sides of the triangle ΔABC to obtain X' and X''. Then we find the endpoints of the inscribed triangle ΔXYZ by intersecting the line through X', X'' with the sides of the triangle ΔABC.



The aim is to **write a mathematical text** containing the statement and a proper proof based on the proposed solution.

A **little reflection** before we start with the writing: The problem assumes implicitly **the existence** of a triangle with smallest perimeter. This is not always guaranteed.

Here is a geometric problem **with no solution**:

Let $A > 0$ be a positive number. Find a rectangle with maximal perimeter amongst all rectangles with area A.

The existence of a triangle with minimal perimeter needs concepts we do not have available at this moment, but here is a thought showing that we cannot find triangles ΔXYZ with arbitrarily small perimeters: If $\mathrm{perim}(\Delta XYZ) = c > 0$, then X has at most distance c to any of the three sides of the triangles ΔABC. But the point with smallest distance to all three sides of ΔABC is the centre of the incircle. Therefore, if $r > 0$ is the radius of the incircle, we always have $\mathrm{perim}(\Delta XYZ) \geq r$.

Now we finished our reflections and start to write the mathematical text.

We first need to **fix notation**:

- $\overline{AB}$ denotes the straight line segment between the points A and B.
- AB denotes the infinite line through two different points A and B.
- $|\overline{AB}|$ denotes the length of the line segment $\overline{AB}$.
- The *perimeter* of a triangle ΔABC is given by

$$\text{perim}(\Delta ABC) = |\overline{AB}| + |\overline{BC}| + |\overline{CA}|.$$

Next we like to give a proper definition of a reflection along a line:

Definition. *Let AB be a line and P a point. Then the reflection $s_{AB}(P)$ of P along AB is defined as*

$$s_{AB}(P) = \begin{cases} P & \text{if } P \in AB \\ P' & \text{if } P \notin AB, \end{cases}$$

where P and P' lie on opposite sides of AB and PP' is perpendicular to AB with

$$|\overline{PX}| = |\overline{XP'}|,$$

where $X = AB \cap PP'$.

Now we have all important notions introduced and can formulate the statement:

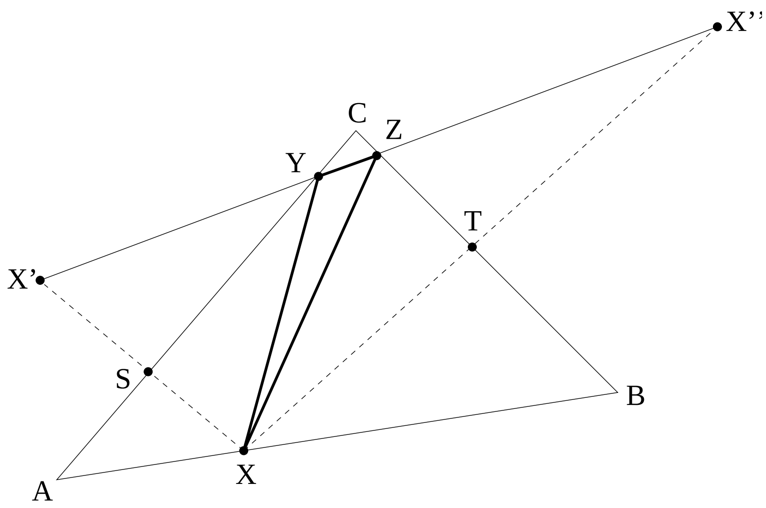
Proposition. *Let ΔABC be an acute triangle. Let $X \in \overline{AB}$ be fixed. Let $X' = s_{AC}(X)$ and $X'' = s_{BC}(X)$. Let $Y = X'X'' \cap AC$ and $Z = X'X'' \cap BC$. Then we have*

$$\text{perim}(\Delta XYZ) \leq \text{perim}(\Delta XY'Z')$$

for all $Y' \in \overline{AC}$ and $Z' \in \overline{BC}$.

We note (without proof) that the acuteness of ΔABC guarantees that the points Y and Z in the proposition lie on the sides $\overline{AC}$ and $\overline{BC}$.

Now we give a proof of the proposition. A crucial observation is that $\overline{XY}$ and $\overline{X'Y}$ are of the same length. We conclude this by introducing the intersection point $S = XX' \cap AC$ and looking at the triangles ΔSXY and $\Delta SX'Y$ (see illustration).

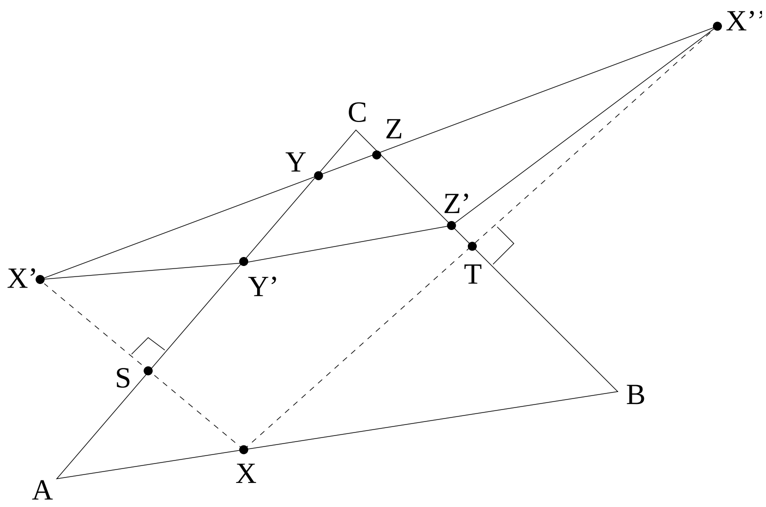


We will use the following facts without proof:

- The shortest path between two different points A, B is the line segment $\overline{AB}$.
- Any two triangles agreeing in two sides and the angle between them are congruent, in which case they agree in all three sides.

Proof. Let $Y' \in \overline{AC}$ and $Z' \in \overline{BC}$. Since the shortest path between X', X'' is the straight line segment, we have

$$|\overline{X'X''}| \leq |\overline{X'Y'}| + |\overline{Y'Z'}| + |\overline{Z'X''}|. \tag{1}$$

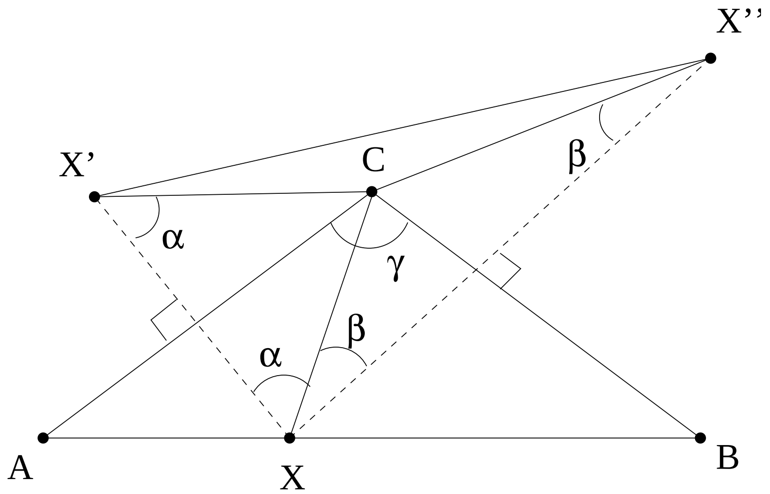


The triangle $\Delta X'Y'S$ is congruent to $\Delta XY'S$ ($|\overline{XS}| = |\overline{X'S}|$ and $\overline{SY'}$ is a common side of both triangles and both triangles are right angled at S). Therefore, we have $|\overline{X'Y'}| = |\overline{XY'}|$ and, similarly $|\overline{X''Z'}| = |\overline{XZ'}|$. Using this fact, we deduce from (1)

$$\begin{aligned} \text{perim}(\Delta XYZ) &= |\overline{XY}| + |\overline{YZ}| + |\overline{ZX}| \\ &= |\overline{X'Y}| + |\overline{YZ}| + |\overline{ZX''}| = |\overline{X'X''}| \\ &\leq |\overline{X'Y'}| + |\overline{Y'Z'}| + |\overline{Z'X''}| \\ &= |\overline{XY'}| + |\overline{Y'Z'}| + |\overline{Z'X}| \\ &= \text{perim}(\Delta XY'Z'). \end{aligned}$$

□

A few words, why acuteness of the triangle ΔABC is important (there is probably **no time to discuss this during the lecture**): Having acute angles at A and B guarantees that the reflection points $X' = s_{AC}(X)$ and $X'' = s_{BC}(X)$ lie in the same halfspace bounded by the infinite line AB like the point C. This is important that the intersection points Y, Z lie also in this halfspace. Moreover, we can show that if ΔABC has an acute angle at C then the line $\overline{X'X''}$ cannot lie outside the triangle ΔABC: Assume that $\overline{X'X''}$ lies outside the triangle ΔABC. Then we have the following situation:



Looking at the angle sum of the triangle $\Delta XX'X''$, we have $2\alpha + 2\beta \leq 180^0$, i.e., $\alpha + \beta \leq 90^o$. Since the angle sum in a quadrilateral is 360^o, we conclude that

$$\gamma = 360^o - 2 \cdot 90^o - (\alpha + \beta) = 180^o - (\alpha + \beta) \geq 90^o.$$

This shows that if $\overline{X'X''}$ lies outside the triangle ΔABC then the angle γ is obtuse. Consequently, if ΔABC is an acute triangle, the line segment $\overline{X'X''}$ intersects the triangle ΔABC.

Finally, let us mention the raise awareness about the following issues when dealing with geometric problems.

Important: Be careful in geometric proofs that

- you consider all possible cases,
- your sketches reflect really occuring situations.