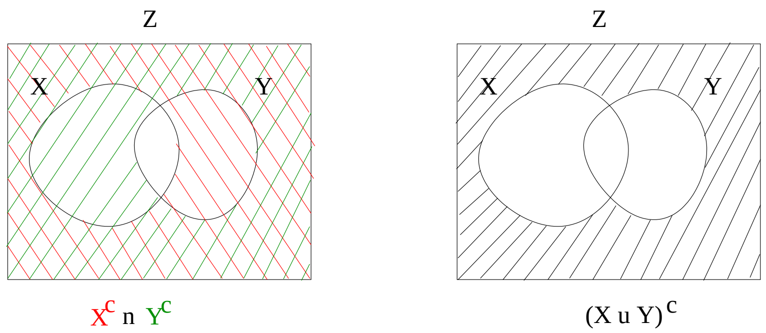
Lecture 4

In the last lecture we introduced mathematical structure elements and discussed how to write a proper mathematical text. In this lecture, we will continue with a bit more logic, in particular **quantifiers and negations** *and with* **infinite unions and intersections of sets***.*

First, recall *De Morgan's Rule* for statements:

$$\text{not } (A \text{ or } B) \quad \Leftrightarrow \quad (\text{not } A) \text{ and } (\text{not } B).$$

There is an analogous statement for sets: Assuming $X, Y \subset Z$. Then we have $(X \cup Y)^c = X^c \cap Y^c$. Here is the Venn Diagram for illustration:



For the proof that both sets are equal, we use arguments from logic:

$$\begin{aligned}
x \in (X \cup Y)^c &\Leftrightarrow x \in Z \backslash (X \cup Y) \\
&\Leftrightarrow x \in Z \text{and not}(x \in X \cup Y) \\
&\Leftrightarrow x \in Z \text{and not}(x \in X \text{ or } x \in Y) \\
&\Leftrightarrow x \in Z \text{ and } [(\text{not } x \in X) \text{ and } (\text{not } x \in Y)] \qquad (1) \\
&\Leftrightarrow (x \in Z \text{ and } (x \notin X)) \text{ and } (x \in Z \text{ and } (x \notin Y)) \\
&\Leftrightarrow (x \in Z \text{ and } (x \notin X)) \text{ and } (x \in Z \text{ and } (x \notin Y)) \\
&\Leftrightarrow (x \in X^c) \text{ and } (x \in Y^c) \\
&\Leftrightarrow x \in X^c \cap Y^c.
\end{aligned}$$

Note that we used De Morgan's Rule at (1).

In fact, the expression "$x \in (X \cup Y)^c$" is a so-called *open statement*, since its truth value can only be determined once we choose an explicit object for x and specify the sets X and Y explicitly. We consider x, X, Y as variables of the open statement which can be specified. Open statements contain those variables and become proper statements, once these variables are explicitly chosen. Logic works also for open statements. Here are examples of other "open statements":

$$\begin{aligned} A(n) &\quad \text{means} \quad n \text{ is a square of an integer}, \\ B(x,y) &\quad \text{means} \quad x^2+y^2 \geq 1, \\ C(x,X,Y) &\quad \text{means} \quad x \in X \backslash Y. \end{aligned}$$

Then the statements $A(4)$, $B(1,-1)$, $C(1/2,\mathbb{Q},\mathbb{N})$ are true, and the statements $A(\pi)$, $B(0,0)$ are false.

Next, we introduce a bit more of notation, **Quantifiers**:

- $\forall$ stands for "for all"
- $\exists$ stands for "there exists"
- $\exists!$ stands for "there exists a unique"

Quantifiers can be combined nicely with open statements.

Example:

$$\forall a \in \mathbb{R}\backslash\{0\} \quad \forall b \in \mathbb{R} \quad \exists! x \in \mathbb{R}: \quad ax = b.$$

This is a true statement, since the unique solution of $ax = b$ is $x = b/a \in \mathbb{R}$. Note that we need here to rule out that $a = 0$.

Be aware: The order of quantifiers matters. The interchange of quantifiers can change completely the meaning of a statement.

Example:

$$\forall a \geq 0 \quad \exists b \in \mathbb{R}: \qquad b^2 = a.$$

This statement means: For all $a \geq 0$ there exists $b \in \mathbb{R}$ such that $b^2 = a$. This is obviously a TRUE statement.

But

$$\exists b \in \mathbb{R} \quad \forall a \geq 0: \qquad b^2 = a$$

means: There exists $b \in \mathbb{R}$ such that for all $a \geq 0$ we have $b^2 = a$. Here the claim is that there exists a universal real number b whose square coincides with every non-negative real number a, which is obviously FALSE.

Now, we want to discuss a method to *negate a statement containing quantifiers.* We start with an illustrative example: We consider the statement

$$\forall z \in \mathbb{C} : z\bar{z} \in \mathbb{R}.$$

This is obviously a TRUE statement. If we would like to show that this statement is false, we would need to find a complex number z with $z\bar{z} \notin \mathbb{R}$. Therefore, the *negation of this statement* claims that there exists a number $z \in \mathbb{C}$ with $z\bar{z} \notin \mathbb{R}$:

$$\exists z \in \mathbb{C} : z\bar{z} \notin \mathbb{R}.$$

The general rules for *negating statement containing quantifiers* are as follows:

(a) Replace $\forall$ by $\exists$ and replace $\exists$ by $\forall$. (Careful: this does not work for $\exists!$)

(b) Negate the conclusion.

Examples:

(a) Statement A: "$\forall a \geq 0 \ \ \exists b \in \mathbb{R} : \qquad b^2 = a$"

Statement (not A): "$\exists a \geq 0 \ \ \forall b \in \mathbb{R} : \qquad b^2 \neq a$",
meaning: "There exists $a \geq 0$ such that we have for all $b \in \mathbb{R}$: $b^2 \neq a$."

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a given function.

Statement B: "$\forall x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2) : \quad x_1 = x_2$",
meaning: "For all $x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2)$, we have $x_1 = x_2$." In short: f is injective.

Statement (not B): "$\exists x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2) : \quad x_1 \neq x_2$",
meaning: "There exist $x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2)$ satisfying $x_1 \neq x_2$." In short: There are two different $x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2)$, i.e., f is not injective.

Finally, we introduce arbitrary unions and intersections. Let us start with a notation for the intersection of finitely many sets with indices: We write

$$\begin{aligned} Z_1 \cup Z_2 \cup \cdots \cup Z_k &= \bigcup_{j \in \{1,2,\ldots,k\}} Z_j, \\ Z_1 \cap Z_2 \cap \cdots \cap Z_k &= \bigcap_{j \in \{1,2,\ldots,k\}} Z_j. \end{aligned}$$

Here the index set is $X = \{1, 2, \ldots, k\}$, but it can be anything, for example $X = \{a, b, c\}$, and then we have

$$\bigcup_{x \in X} Z_x = Z_a \cup Z_b \cup Z_c.$$

Note that an element in $\bigcap_{x \in X} Z_x$ must belong to each of the sets Z_x, while an element in $\bigcup_{x \in X} Z_x$ must only belong to at least one of the sets Z_x, i.e.,

$$\bigcup_{j \in \{1,2,\ldots,k\}} Z_j = \{z \mid \exists j \in \{1, 2, \ldots, k\} : z \in Z_j\}, \qquad (2)$$

$$\bigcap_{j \in \{1,2,\ldots,k\}} Z_j = \{z \mid \forall j \in \{1, 2, \ldots, k\} : z \in Z_j\}. \qquad (3)$$

The description of $\bigcup Z_x$ and $\bigcap Z_x$ with $\exists, \forall$-quantifiers in (2) and (3) allows even to extend the definition to infinite index sets. We generally define for finite and infinite index sets X, indexing a family of sets Z_x, the union and intersection of these sets by

$$\bigcup_{x \in X} Z_x = \{z \mid \exists x \in X : z \in Z_x\},$$

$$\bigcap_{x \in X} Z_x = \{z \mid \forall x \in X : z \in Z_x\}.$$

Examples:

- $\bigcup_{k \in \mathbb{Z}} (k, k+1) = \mathbb{R} \backslash \mathbb{Z}$: Every $x \in \mathbb{R} \backslash \mathbb{Z}$ lies in one of the intervals $(k, k+1)$. On the other hand, the integers do not lie in this union, since none of these intervals contains an integer.

- $\bigcap_{p \text{ prime}} \{kp \mid k \in \mathbb{N}\} = \emptyset$, since every natural number n has only finitely many primes dividing it, but there are infinitely many primes.

Assume Z_x are sets indexed by $x \in X$, and that the sets Z_x are all subsets of a larger set Z, so that we can take complements in Z. Then we have the following "general" De Morgan Rule for arbitrarily many sets:

$$\left(\bigcap_{x \in X} Z_x \right)^c = \bigcup_{x \in X} Z_x^c.$$

Here is the proof:

$$
\begin{aligned}
\left(\bigcap_{x\in X} Z_x\right)^c &= \{z \in Z \mid \forall x \in X : z \in Z_x\}^c \\
&= \{z \in Z \mid \text{not}(\forall x \in X : z \in Z_x)\} \\
&= \{z \in Z \mid \exists x \in X : z \notin Z_x\} \\
&= \{z \mid \exists x \in X : (z \in Z \text{ and } z \notin Z_x)\} \\
&= \{z \mid \exists x \in X : z \in Z_x^c\} \\
&= \bigcup_{x\in X} Z_x^c.
\end{aligned}
$$