Lecture 9

In the last lecture we presented ideas by Georg Cantor *to compare the size of finite and infinite sets. In this lecture we introduce the important notions of* **preimage and image of a map** *and of* **equivalence relations and equivalence classes**.

In the meantime, you should be somewhat familiar with the concepts of injectivity, surjectivity and bijectivity. If a map $f : X \to Y$ is bijective, we can define its inverse $f^{-1} : Y \to X$. But even if f is not injective, there is a way to define $f^{-1}(Y_0)$ for sets $Y_0 \subset Y$. $f^{-1}(Y_0)$ is again a set, namely, a subset of X, containing all elements which are mapped into Y_0.

Definition. *Let $f : X \to Y$ be a map, not necessarily injective. Then the* **preimage** *of a set $Y_0 \subset Y$ is defined as the set*

$$f^{-1}(Y_0) := \{x \in X \mid f(x) \in Y_0\} \subset X,$$

i.e., as the set of all elements in X whose images under f lie in Y_0. Moreover, the **image** *of a set $X_0 \subset X$ is defined by*

$$f(X_0) := \{f(x) \mid x \in X_0\} \subset Y,$$

i.e., the set of all images of elements in X_0.

We obviously have $f^{-1}(Y) = X$ and, if $f : X \to Y$ is bijective, $f^{-1}(Y_0) = \{f^{-1}(y) \mid y \in Y_0\}$, where the left hand f^{-1} is the preimage of f and the right hand f^{-1} is the inverse map $f^{-1} : Y \to X$.

Examples: (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Since $f(-1) = f(1) = 1$, the function f is not injective. But we have

$$\begin{aligned} f^{-1}(\{3\}) &= \{-\sqrt{3}, \sqrt{3}\}, \\ f^{-1}(\{0\}) &= \{0\}, \\ f^{-1}(\{-1\}) &= \{\}, \\ f^{-1}((1,10]) &= [-\sqrt{10}, -1) \cup (1, \sqrt{10}], \\ f^{-1}((-1,4)) &= (-2,2). \end{aligned}$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \sin(x)$. Then we have

$$\begin{aligned} g^{-1}(\{0\}) &= \{k\pi \mid k \in \mathbb{Z}\}, \\ g^{-1}([-1,1]) &= \mathbb{R}, \\ g^{-1}([0,1]) &= \bigcup_{k \in \mathbb{Z}} [2k\pi, (2k+1)\pi]. \end{aligned}$$

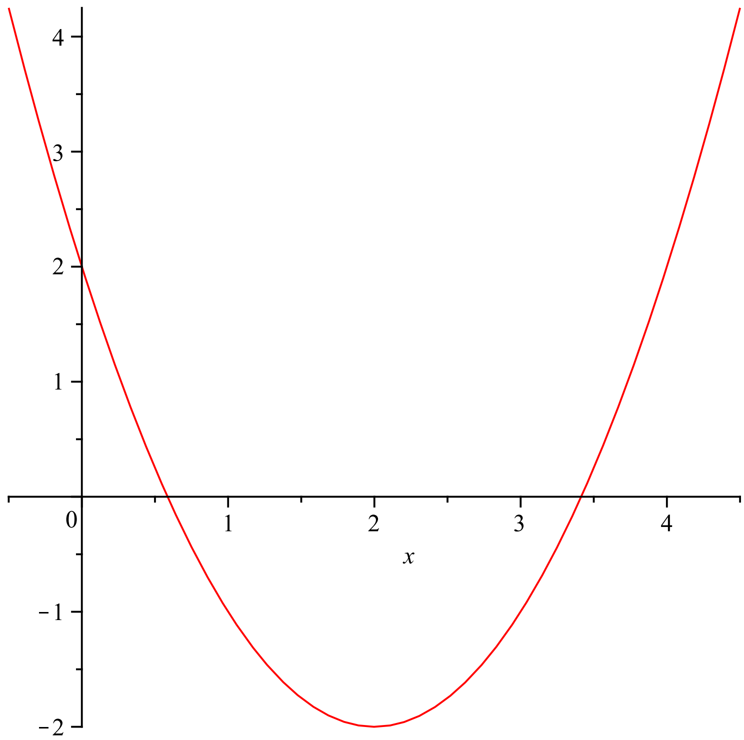
(c) Let $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2 - 4x + 2$. Then the preimage $h^{-1}((-\infty, 2])$ is

$$h^{-1}((-\infty, 2]) = \{x \in \mathbb{R} \mid h(x) \leq 2\}.$$

$h(x) \leq 2$ translates to $x^2 - 4x \leq 0$, i.e., $x(x-4) \leq 0$, which implies

$$\{x \in \mathbb{R} \mid h(x) \leq 2\} = [0, 4].$$

The following picture illustrates the preimage $h^{-1}((-\infty, 2])$ as the set of points of the real axis whose images are ≤ 2.



Next, we introduce the notion of an **equivalence relation**. The crucial properties of the equality relation "=" are the following:

(a) $x = x$,

(b) if $x = y$ then $y = x$,

(c) if $x = y$ and $y = z$ then $x = z$.

Sometimes, we want to introduce a weaker relation than equality on the elements of a set X, but these crucial properties should still hold. Such relations are called equivalence relations. Here is the precise definition.

Definition. *A relation* $\sim$ *on a set* X *is called an* **equivalence relation** *if the following three conditions are satisfied:*

(a) $x \sim x$ *(reflexive condition),*

(b) if $x \sim y$ *then* $y \sim x$ *(symmetric condition),*

(c) if $x \sim y$ *and* $y \sim z$ *then* $x \sim z$ *(transitive condition).*

Obviously, equality is an equivalence relation on any set. Here is another example:

Example: Let $n \in \mathbb{N}$. We say that two integers $x, y \in \mathbb{Z}$ are equivalent and write $x \sim y$, if x and y have the same remainder under division by n. This can also be rephrased by the condition that $x - y$ is divisible by n. (If $n = 3$, we have $4 \sim 7$ and $16 \sim -2$.) Let us check Transitivity: Let $x \sim y$ and $y \sim z$. Then we know that n divides $x - y$ and $y - z$ and, therefore, also $(x - y) + (y - z) = x - z$. This shows that $x \sim z$.

The relation $\geq$ is **not** an equivalence relation: We have Reflexivity: $x \geq x$, and Transitivity: $x \geq y$ and $y \geq z$ imply $x \geq z$, but Symmetry is violated: $3 \geq 2$ but not $2 \geq 3$.

Once, we have a set with an equivalence relation, we can define the equivalence classes:

Definition. *Let* X *be a set and* $\sim$ *be an equivalence relation on* X*. The* **equivalence class** *of* $x \in X$*, denoted by* $[x]$*, is the set*

$$[x] = \{y \in X \mid y \sim x\}.$$

The element $x \in X$ *is called a* **representative** *of the equivalence class* $[x]$*.*

Example: With the equivalence relation of the previous example and $n = 3$ we have

$$\begin{aligned} [0] &= \{\ldots, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\}, \\ [1] &= \{\ldots, -8, -5, -2, 1, 4, 7, 10, 13, \ldots\}, \\ [2] &= \{\ldots, -7, -4, -1, 2, 5, 8, 11, 14, \ldots\}, \end{aligned}$$

but also $[5] = \{\ldots, -1, 2, 5, 8, 11, \ldots\} = [2]$. We see that the representatives 2 and 5 have the same equivalence classes.

In view of this example, it seems to be true that two equivalence classes are either disjoint or they agree completely. This is generally true and is formulated in the following theorem.

Theorem. *Let X be a set and $\sim$ be an equivalence relation. Then we have for the equivalence classes of any two elements $x_1, x_2 \in X$: Either both equivalence classes agree ($[x_1] = [x_2]$) or they are disjoint ($[x_1] \cap [x_2] = \emptyset$). Moreover, we have*

$$X = \bigcup_{x \in X} [x],$$

and the equivalence classes form a partition of the whole set X into pairwise disjoint subsets.

Proof. We prove that if $x_1 \sim x_2$ then $[x_1] = [x_2]$: By the Symmetry property, it suffices to prove

$$x_1 \sim x_2 \quad \Rightarrow \quad [x_1] \subset [x_2].$$

Assume $x_1 \sim x_2$ and $y \in [x_1]$. Then we have $y \sim x_1$. Since $x_1 \sim x_2$, we conclude from Transitivity that $y \sim x_2$. But this means that $y \in [x_2]$.

Next we prove that if $x_1 \not\sim x_2$ then $[x_1] \cap [x_2] = \emptyset$: We use the Contrapositive method and prove

$$[x_1] \cap [x_2] \neq \emptyset \quad \Rightarrow \quad x_1 \sim x_2,$$

instead. If $[x_1] \cap [x_2] \neq \emptyset$, then there exists $y \in X$ with $y \in [x_1] \cap [x_2]$, i.e., $y \sim x_1$ and $y \sim x_2$. Using Symmetry and Transitivity, we conclude that $x_1 \sim y$ and $y \sim x_2$ and, therefore $x_1 \sim x_2$.

Since $x \in [x]$, by Reflexivity, we see that $X = \bigcup_{x \in X} [x]$. The subsets $[x] \subset X$ are either disjoint or agree, by the above arguments, so they partition the set X into pairwise disjoint subsets. $\square$

Finally, let us look at another example:

Example: Let $f : X \to Y$ be a map. It is easy to see that then the relation

$$x_1 \sim x_2 \quad \Rightarrow \quad f(x_1) = f(x_2)$$

defines an equivalence relation. The equivalence classes are precisely the preimages, i.e.,

$$[x] = f^{-1}(\{f(x)\}).$$

In the case $f : \mathbb{R}^2 \to \mathbb{R}$, $f(a,b) = a^2 + b^2 = \|(a,b)\|^2$, two points $(a,b), (c,d) \in \mathbb{R}^2$ are equivalent if they have the same distance to the origin $(0,0)$ and the equivalence classes are the (pairwise disjoint) circles around the origin in $\mathbb{R}^2$.