Lecture 10

In this lecture we will reflect on the **first two take home assessments**.

Here are some comments on the first problem of the First Take Home Assessment.

It was asked for a rigorous proof of the inclusion

$$X \backslash Y \subset (X \backslash Z) \cup (Z \backslash Y). \tag{1}$$

Be aware that the Venn Diagram can illustrate the correctness of the inclusion but does not substitute a rigorous proof. So you need to use a certain amount of creativity and ingenuity whenever you are faced with the task to prove a concrete inclusion or an equality of sets.

It was mentioned in an earlier lecture that an inclusion $X \subset Y$ can be proved by showing that every element of X is also an element of Y. The first solution is based on this approach:

Let $x \in X \backslash Y$. So we know that $x \in X$ and $x \notin Y$. There is a third set on the right hand side of (1), namely, the set Z. So it makes sense to ask whether $x \in Z$ or $x \notin Z$. If we show that both cases lead to

$$x \in (X \backslash Z) \cup (Z \backslash Y),$$

we have proved (1).

If $x \in Z$, we obviously have $x \in Z \backslash Y$, since we know that $x \notin Y$. This obviously implies that

$$x \in (X \backslash Z) \cup (Z \backslash Y).$$

If $x \notin Z$, we obviously have $x \in X \backslash Z$, since we know that $x \in X$. This obviously implies that

$$x \in (X \backslash Z) \cup (Z \backslash Y),$$

finishing the proof of (1).

There is usually not only one way to prove a mathematical statement. I want to present another proof of (1): First of all introduce the set $W = X \cup Y \cup Z$. Since $X, Y, Z \subset W$, we can take their complements in W. Therefore, we can replace $X \backslash Y$ by $X \cap Y^c$. Similarly, we have $X \backslash Z = X \cap Z^c$ and $Z \backslash Y = X \cap Y^c$. So (1) is equivalent to

$$X \cap Y^c \subset (X \cap Z^c) \cup (Z \cap Y^c).$$

Since $Z \cup Z^c = W$, we have

$$\begin{aligned} X \cap Y^c &= (X \cap Y^c) \cap W \\ &= (X \cap Y^c) \cap (Z \cup Z^c) \\ &= (X \cap Y^c \cap Z) \cup (X \cap Y \cap Z^c). \end{aligned}$$

We used here the fact that if $U \subset V$ then $U = U \cap V$, which has a very easy proof. (If $U \subset V$ and $x \in U$, we also have $x \in V$ and, therefore $x \in U \cap V$. Conversely, if $x \in U \cap V$, then $x \in U$.) Moreover, we used fundamental laws, like distributivity and associativity. Using

$$U \cap V \subset U,$$

which has a similarly easy proof (if $x \in U \cap V$, then $x \in U$ and $x \in V$), we conclude that $X \cap Y^c \cap Z \subset Y^c \cap Z$ and $X \cap Y \cap Z^c \subset Y \cap Z^c$ and, therefore, by the law of commutativity,

$$X \cap Y^c = (X \cap Y^c \cap Z) \cup (X \cap Y \cap Z^c) \subset (Z \cap Y^c) \cup (X \cap Z^c) = (X \cap Z^c) \cup (Z \cap Y^c),$$

finishing the proof.

A third approach to prove (1) might be algorithmically via truth tables. But be aware that truth tables deal with statements. So the following "truth table" makes NO SENSE at all:

X	Y	Z	$X\backslash Y$	$X\backslash Z$	$Z\backslash Y$	$(X\backslash Z) \cup (Z\backslash Y)$
true	true	true	false	false	false	false
true	true	false	false	true	false	true
true	false	true	true	false	true	true
true	false	false	true	true	false	true
false	true	true	false	false	false	false
false	true	false	false	false	false	false
false	false	true	false	false	true	true
false	false	false	false	false	false	false

However, replacing the top line by the open statements "$x \in X$", "$y \in X$", "$x \in Z$", "$x \in X\backslash Y$", "$x \in X\backslash Z$", "$x \in Z\backslash Y$", "$x \in (X\backslash Z) \cup (Z\backslash Y)$" leads to a proper truth table. But then the proof is not finished. You would then have to check that whenever $x \in X\backslash Y$ is true then also $x \in (X\backslash Z) \cup (Z\backslash Y)$ is true. This would lead to a complete proof.

Personally, I am not too keen on the third proof, since there is no elegance in it, it is just tedious "book keeping" by listing every possible case.

On a similar note, let me mention that there are short and lengthy proofs of mathematical facts. If you found a proof, it is often worthwile to ponder whether there is an easier way to derive the statement. It is then often helpful to leave the question alone and return to it after a break with a fresh mind. Obviously, a short elegant proof is preferable to a lengthy and difficult proof. On the other hand, the crucial step in solving a mathematical problem is first to find *some solution*. This is really essential! Only then it makes sense to think about improvements/simplications or alternative solutions.

Now let us have a closer look at part (a) of the second problem of the Second Take Home Assessment.

It is assumed that $a > 0$ and $u_1 > a$. The sequence $u_1, u_2, \ldots$ satisfies the recurrency relation

$$u_{n+1} := \frac{1}{2}\left(u_n + \frac{a^2}{u_n}\right) \quad \forall n \in \mathbb{N}.$$

You should prove that $\lim_{n\to\infty} u_n = a$. Splitting this up leads to the following two tasks:

(a) Showing that (u_n) has a limit.

(b) Showing that the limit is a.

When you start your solution by writing "Let $L = \lim_{n\to\infty} u_n$", you skipped the existence proof of the limit, which MUST COME FIRST. For example, consider a sequence (x_n) satisfying the recurrence relation $x_{n+1} = \frac{1}{2}(x_n^2 + 1)$. The approach $L = \lim_{n\to\infty} x_n$ would lead to

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}(x_n^2 + 1) = \frac{1}{2}(L^2 + 1),$$

i.e., $L^2 - 2L + 1 = 0$, i.e., $L = 1$. But it would be wrong to conclude that the sequence (x_n) is convergent to the limit 1. We easily see that $x_n \geq 2$ would imply

$$x_{n+1} = \frac{1}{2}(x_n^2 + 1) \geq \frac{1}{2} x_n x_n \geq x_n.$$

This would imply (by Induction) that (x_n) is monotone increasing for the start value $x_1 = 2$ and, therefore, would never converge to 1.

So, assuming $L = \lim_{n\to\infty} u_n > 0$ and deriving

$$L = \lim u_{n+1} = \frac{1}{2} \lim u_n + \frac{a^2}{u_n} = \frac{1}{2}(L + \frac{a^2}{L})$$

yields only: In case, (u_n) has a positive limit L, then this limit must satisfy $L^2 = a^2$. Obviously, $L^2 = a^2$ and the assumptions $L, a > 0$ would then imply $L = a$. But these calculations give NO GUARANTEE at all that (u_n) has a limit.

You can conclude that (u_n) has a limit if you show that (u_n) is monotone decreasing and bounded from below by a, using a standard result from the Analysis course. This would provide a proper proof of Task (a). But be alert: This does not imply that the sequence (u_n) must converge TOWARDS THE LOWER BOUND a. For example, the sequence $x_n = 2 + \frac{1}{n}$ is monotone decreasing and has lower bound $a = 1$, but we have

$$\lim_{n\to\infty} x_n = 2 \neq 1 = a.$$

So once you have finished Task (a), you still need to identify the limit.

It was proposed that you might prove that $u_n > a$ implies both $u_{n+1} > a$ and $u_{n+1} - a < \frac{1}{2}(u_n - a)$. These implications per se have nothing to do with Induction. Once you have established both implications, you may use them in an Induction Proof that you have $0 < u_n - a < \frac{1}{2^{n-1}}(u_1 - a)$ for all $n \geq 2$. Then the Squeezing Theorem implies that $u_n \to a$, and you have reached your goal IN ONE GO, and without splitting it up into two separate tasks.

Be aware that, when presenting a sequence of formulas, you MUST explain how each formula is related to the previous one. For example, the sequence of formulas

$$\begin{aligned} u_n &> a > 0, && && (2)\\ u_n^2 &> a^2, && \text{(taking squares)}, && (3) \end{aligned}$$

does not tell the reader whether you mean that "(2) implies (3)" or "(3) implies (2)", or even "(2) is equivalent to (3)". On the other hand, the following text is unambigous: "We conclude from

$$u_n > a > 0$$

that

$$u_n^2 > a^2,$$

by taking squares." So the text explains your thought process.

Here is a proof of the Hint given in this problem: Assume that $u_n > a > 0$. Then $\frac{1}{u_n}$ is well defined and we have

$$u_{n+1} - a = \frac{1}{2}\left(u_n + \frac{a^2}{u_n}\right) - a = \frac{1}{2u_n}(u_n^2 + a_2 - 2au_n) = \frac{1}{2}\frac{u_n - a}{u_n}(u_n - a). \quad (4)$$

Since $0 < u_n - a < u_n$, we have $0 < \frac{u_n - a}{u_n} < 1$, and we conclude from $0 < u_n - a < u_n$ and (4) that

$$0 < u_{n+1} - a = \frac{u_n - a}{u_n} \frac{1}{2}(u_n - a) < \frac{1}{2}(u_n - a) \tag{5}$$

for all $n \in \mathbb{N}$.

Here is a proper **Induction Proof** that $u_n - a \leq \frac{1}{2^{n-1}}(u_1 - a)$ for all $n \geq 2$: Let $A(n)$ be the open statement "$0 < u_n - a < \frac{1}{2^{n-1}}(u_1 - a)$".

Start: Choosing $n = 1$ in (5) gives

$$0 < u_2 - a < \frac{1}{2}(u_1 - a),$$

proving $A(2)$.

Induction Step: Assume that $A(n)$ holds **for a certain** $n \in \mathbb{N}$, $n \geq 2$. (Do never write "Assume that $A(n)$ holds **for all** $n \in \mathbb{N}$, $n \geq 2$". Then you would assume that everything you want to prove is already true. This does NOT MAKE SENSE AT ALL and indicates that you do not understand yet the philosophy behind Induction!) Then we have

$$0 < u_n - a < \frac{1}{2^{n-1}}(u_1 - a).$$

Using (5) and the assumption $u_n - a > 0$, we conclude that

$$0 < u_{n+1} - a < \frac{1}{2}(u_n - a) < \frac{1}{2^n}(u_1 - a).$$

This shows that $A(n)$ implies $A(n+1)$. $\square$

Once you have established

$$0 < u_n - a < \frac{1}{2^{n-1}}(u_1 - a)$$

for all $n \in \mathbb{N}$, $n \geq 2$, you can conclude that $u_n \to a$, by the Sqeezing Theorem, since $\frac{1}{2^n} \to 0$.