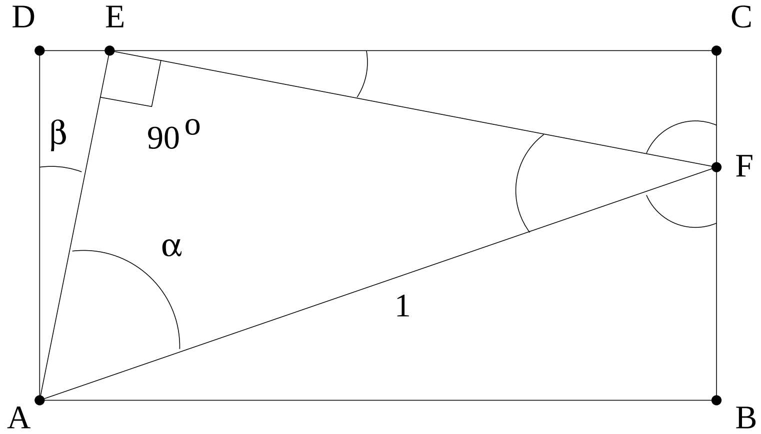
Writing Maths Problems (Week 3)

Question 1 Let $\alpha, \beta \geq 0$ be two angles with $\alpha + \beta < 90°$. Consider the following diagram in the rectangle $\square ABCD$.



(a) Find all angles and the lengths of all the line segments in the picture in terms of the angles α and β.

(b) What fundamental facts for trigonometric functions are proved with this diagram?

Question 2 Give a geometric proof of the following result. Start with sketches and decide which additional lines might be helpful for your arguments. Make sure that you cover every possible geometric case.

Let A, B, C be three distinct points on a circle with centre M. Assume that C and M lie on the same side of the infinite straight line AB. Then we have

$$\measuredangle AMB = 2\measuredangle ACB.$$

Once you collected your arguments, write the proof as a proper mathematical text with appropriate illustrations. You may use the following facts without proof, but refer to them clearly whenever you use them.

(1) The sum of angles in a triangle is $180°$.

(2) The base angles of an isosceles triangle are equal.

Question 3 (Jigsaw Puzzle 1) The task is to transform the following jigsaw puzzle into a logically correct mathematical text. Here are the pieces to be arranged into a logical order:

1. **Definition**

2. **Theorem**

3. **Proof**

4. Subtracting $(*)$ from $(\diamond)$, and observing that most terms cancel out, leads to

5. $$S_n(q) = \frac{q^n - 1}{q - 1}. \qquad (\Box)$$

6. Let $q \in \mathbb{R}$. We call

7. Multiplication of $(*)$ with q gives

8. $$qS_n(q) - S_n(q) = (q-1)S_n(q) = q^n - 1.$$

9. the geometric series of q of length n.

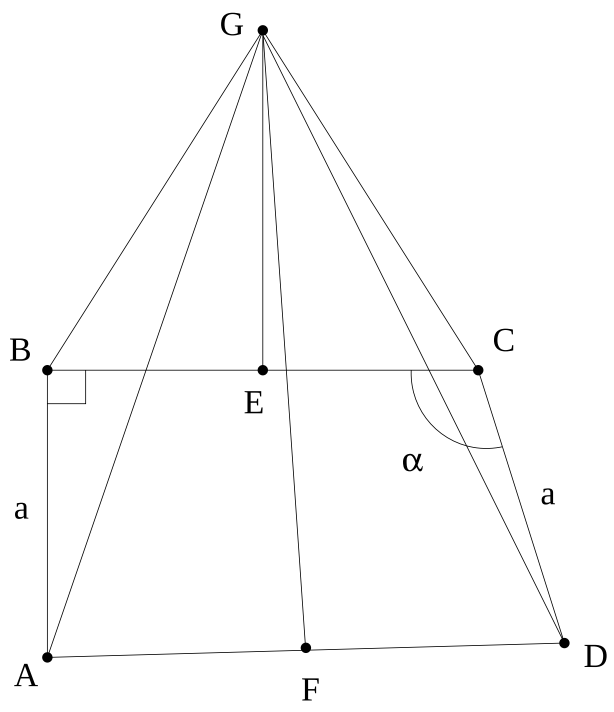
10. Let $q \neq 1$. Then we have

11. $$S_n(q) = 1 + q + q^2 + \cdots + q^{n-1} \qquad (*)$$

12. Now, division by $(q-1) \neq 0$ yields $(\Box)$, finishing the proof.

13. $$qS_n(q) = q + q^2 + q^3 + \cdots + q^n. \qquad (\diamond)$$

Question 4 The following fake proof is attributed to the author of Alice in Wonderland. Few people know that Charles Dodgson, better known as Lewis Carroll, was also a mathematician (in fact, a logician). Find out what is wrong in the following proof that an obtuse angle is a right angle:



Let $\square ABCD$ be a quadrilateral with a right angle at B and an obtuse angle $\alpha > 90°$ at C. Moreover, let $|\overline{AB}| = |\overline{CD}| = a$. Let G be the intersection point of the perpendicular bisectors of the line segments $\overline{AD}$ and $\overline{BC}$ (since $\overline{AD}$ and $\overline{BC}$ are not parallel, these perpendicular bisectors have an intersection point). The triangles ΔAFG and ΔDFG are congruent, since they agree in two sides and the angle between them ($|\overline{AF}| = |\overline{DF}|$ and $\overline{FG}$ is a common side and both triangles are right-angled at F). This shows that $|\overline{AG}| = |\overline{DG}|$. Similarly, we prove that $|\overline{BG}| = |\overline{CG}|$. Since $|\overline{AB}| = |\overline{CD}|$, by construction, both triangles ΔABG and ΔDCG agree in three sides and are, therefore, congruent. But then $\measuredangle ABG = \measuredangle DCG$ and, since $\measuredangle EBG = \measuredangle ECG$ because the right-angled triangles ΔBEG and ΔCEG are congruent, we conclude that

$$90° = \measuredangle ABE = \measuredangle DCE = \alpha.$$

But this shows that the obtuse angle α is equal to a right angle, which is a contradiction!

Question 5 (Jigsaw Puzzle 2) Here is a more complicated puzzle.

1. **Definition**, **Example**, **Definition**, **Example**, **Theorem**, **Proof**

2. Then the set $S := \{0, 2, 4, 6, 10\} \subset A$ is 1-separating, but not maximally 1-separating, since the bigger set $S' := \{0, 2, 4, 6, 8, 10\} \subset A$ is also 1-separating.

3. Now we present the main result of this note.

4. Let $A \subset \mathbb{R}$ be a subset and $r > 0$. If the finite set $S \subset A$ is a maximally r-separating set,

5. We first introduce the notions of r-separating and r-covering sets.

6. Therefore, the strictly bigger set $S' := \{x_1, \ldots, x_n, x\} \subset A$ would also be r-separating. This is a contradiction to the assumption that S is **maximally** r-separating.

7. if any strictly bigger set $S' \subset A$ of finitely many points is no longer r-separating.

8. Let $A \subset \mathbb{R}$ be a subset and $r > 0$. A finite set $S := \{x_1, \ldots, x_n\} \subset A$ is called *r-separating*,

9. Let A be the closed interval $[0, 10]$.

10. if the open intervals $(x_i - r, x_i + r)$ are pairwise disjoint. A finite r-separating set $S = \{x_1, \ldots, x_n\} \subset A$ is called *maximally r-separating*,

11. if the union of the open intervals $(x_i - r, x_i + r)$ covers the set A.

12. then S is also a $2r$-covering set.

13. Let $B := \{1/n \mid n \in \mathbb{N}\}$. Then the finite set $S := \{1, 1/2, 1/4, 1/8\} \subset B$ is 1/8-covering.

14. Let the finite set $S \subset A$ be given by $\{x_1, \ldots, x_n\}$. Assume S would not be $2r$-covering.

15. Let $A \subset \mathbb{R}$ be a subset and $r > 0$. A finite set $S := \{x_1, \ldots, x_n\} \subset A$ is called *r-covering*,

16. Then we could find a point $x \in A$ which is not in the union of the intervals $(x_i - 2r, x_i + 2r)$. This would mean that x has distance greater or equal to $2r$ to all the points x_i.