Answers to More Logic and Sets Problems

Question 1

1. The long version is: "For all $C > 0$ there exist $n \in \mathbb{N}$ such that we have $x_n > C$." In plain terms: The sequence (x_n) is not bounded from above.

   The negation is:
   $$\exists C > 0 \ \ \forall n \in \mathbb{N} : \quad x_n \leq C.$$
   In plain terms: The sequence (x_n) is bounded from above.

2. The long version is: "There exists $L > 0$ such that for all $x, y \in \mathbb{R}$, we have $|f(x) - f(y)| \leq L|x - y|$." In plain terms: The y-difference between any two points on the graph of f is bounded from above by L times the x-distance of these two points.

   The negation is:
   $$\forall L > 0 \ \ \exists x, y \in \mathbb{R} : \quad |f(x) - f(y)| > L|x - y|.$$
   In plain terms: For every $L > 0$ there are two points on the graph of f such that the y-distance between them is bigger than L times the x-distance between them.

3. The long version is: "For all $y \in Y$ there exists $x \in X$ such that $y = g(x)$." In plain terms: Any element of Y is an image point of g, i.e., g is surjective.

   The negation is
   $$\exists y \in Y \ \ \forall x \in X : \quad y \neq g(x),$$
   or even shorter
   $$\exists y \in Y : \quad y \notin g(A).$$
   In plain terms: The image of g does not cover all of Y.

Question 2

1. $\displaystyle\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1\right) = (0, 1)$. Since every set $[1/n, 1)$ is contained in $(0, 1)$, we have the inclusion "$\subset$". On the other hand, for every $x \in (0, 1)$ we have $x > 0$ and there exists $n \in \mathbb{N}$ such that $1/x < n$, i.e., $1/n < x$. This shows that $x \in [1/n, 1)$ and, therefore, x lies in the union $\bigcup_{n \in \mathbb{N}} [1/n, 1)$. This shows the inclusion "$\supset$", and both sets are equal.

2. $\displaystyle\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n},\frac{2}{n}\right) = \{0\}$. Since $0 \in (-1/n, 2/n)$ for all $n \in \mathbb{N}$, we have the inclusion "$\supset$". We show that there is no real $x \neq 0$ in this intersection. We can find $n \in \mathbb{N}$ such that $|x| > 1/n$. This implies that $x \notin (-1/n, 2/n)$ and, therefore, $x \notin \bigcap_{n\in\mathbb{N}}(-1/n, 2/n)$. This shows the inclusion "$\subset$", and both sets are equal.

3. $\displaystyle\bigcup_{n\in\mathbb{N}}[1,n) = [1,\infty)$. Obviously, we have $[1,n) \subset [1,\infty)$, which shows the inclusion "$\subset$". For every $x \in [1,\infty)$, there exists $n \in \mathbb{N}$ with $x < n$, and we see that $x \in \bigcup_{n\in\mathbb{N}}[1,n)$. This shows the inclusion "$\supset$", and both sets are equal.

4. $\displaystyle\bigcup_{q\in\mathbb{Q}}\left(q-\frac{1}{1000}, q+\frac{1}{1000}\right) = \mathbb{R}$. This follows from the fact that the rational numbers are dense in the set of real numbers. Obviously, we have the inclusion "$\subset$". Now that $x \in \mathbb{R}$. Then there is a rational number q agreeing with x up to the first 5 digits after the decimal point, so we have $|q-x| \le 10^{-5}$ and $x \in (q-1/1000, q+1/1000)$. This shows the inclusion "$\supset$", and both sets are equal.

Question 3

(i) The intersection consists of those subsets of $\{1,2,3,4,5\}$ which contain 1 and not 5 and consist of three numbers. This leads to

$$R \cap S = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}.$$

(ii) The statement is false: Let $X = \mathbb{N}$. Then $X \in \mathcal{P}(\mathbb{N})$ and X has infinitely many elements. If X were an element of the right hand set, it would have to be an element of a power set $\mathcal{P}(\{1,2,\dots,N\})$, for some $N \in \mathbb{N}$. But each of the power sets $\mathcal{P}(\{1,2,\dots,N\})$ has only finite many elements. Therefore, X is not an element on the right hand set.

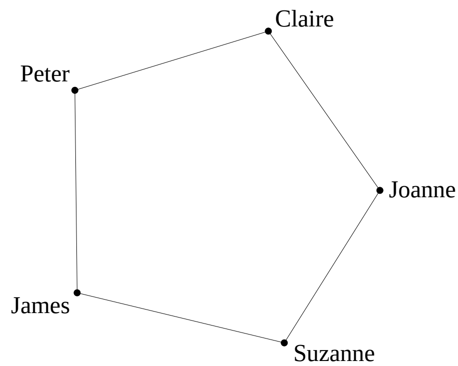
Question 4 We see that the last digits of the powers 2^n follow the repetitive pattern $2,4,8,6,2,4,8,6,\dots$, i.e., we have for all $n \in \mathbb{N}$

$$\text{last digit of } 2^n = \begin{cases} 2 & \text{if divison of } n \text{ by 4 leaves remainder 1,} \\ 4 & \text{if divison of } n \text{ by 4 leaves remainder 2,} \\ 8 & \text{if divison of } n \text{ by 4 leaves remainder 3,} \\ 6 & \text{if } n \text{ is divisible by 4.} \end{cases}$$

This immediately implies equality of both sets.

Question 5 Since every sequence having a limit is bounded, we have $S_2 \subset S_1$. Bounded or convergent sequences do not need to be monotone increasing, so neither S_1 nor S_2 are subsets of S_3 or S_4. On the other hand, the sequence $(x_n = n)$ is monotone increasing, but neither bounded nor convergent, therefore S_3 is not a subset of S_1 or of S_2. S_4 is obviously a subset of S_3. But every monotone increasing sequence of real numbers, bounded from above has a limit convergent, by the completeness of the real numbers. Therefore, we have $S_4 \subset S_2 \subset S_1$.

Question 6 There can be parties with 5 people where it is not true that *at least three people who do not know each other* **or** *there are at least three people who know each other.* To see this, look at the following graph with 5 points representing the people and lines connecting these points if and only if the people know each other. There are obviously now three people knowing each other pairwise. On the other hand, each of the 5 people does not know precisely two people who know each other. Therefore, there are no three people who do not know each other pairwise.



Next, we show that for every party with 6 people we always have *at least three people who do not know each other* **or** *there are at least three people who know each other.* Choose one of them. We call this person Jimmy. Since there are 5 more people at the party, Jimmy knows at least three of them or Jimmy does not know at least three of them. Let us go through both cases:

Case 1: Jimmy knows (at least) three other people: We choose three of them. Either all three of them do not know each other, in which case there are *at least three people who do not know each other*, or there are two of them knowing each other, in which case these two together with Jimmy form a group of three people knowing each other pairwise. This confirms for this case that the required condition is satisfied.

Case 2: Jimmy does not know (at least) three other people: We choose three of them. Either all three of them know each other, in which case there are *at least three people who know each other*, or there are two of them who do not know each other, in which case these two together with Jimmy form a group of three people who do not know each other pairwise. So the required condition is also satisfied in this case.

The upshot is: There are at least 6 people at this party.