Proof Problems (Week 5)

The proof techniques used in the following questions are **Induction** and **Indirect Proof**.

Question 1 Several straight lines are drawn on a plane. Prove that one can colour the regions, created as a result of the intersection of the lines, black and white so that adjacent regions have different colours.

Question 2 Prove the following facts:

(a) Let $x, y \in \mathbb{N}$ and $x > y$. If $2y + 1$ is not a prime then $x^2 - y^2$ is also not a prime.

(b) For any pair of positive integers x, y we have $\sqrt{x^2 + y^2} \neq x + y$.

Question 3 *(taken from M.V. Day: "An Introduction to Proofs and the Mathematical Vernacular")* Show that every integer $n \geq 12$ can be written in the form $n = 7l + 3m$ where l and m are nonnegative integers.

Question 4 *(taken from E.J. Barbeau: "Mathematical fallacies, flaws and flimflam")* For $n \in \mathbb{N}$, let $A(n)$ be the open statement that if k, l are natural numbers with $\max(k, l) = n$ then $k = l$. Here is an Induction Proof that $A(n)$ is true for all $n \in \mathbb{N}$:

Start of Induction ($n = 1$): If k, l are natural number with $\max(k, l) = 1$ then we have $k = l = 1$. This shows that $A(1)$ is true.

Induction Step: Assume that $A(n)$ is true for some $n \in \mathbb{N}$. Let k, l be natural numbers such that $\max(k, l) = n + 1$. Then $\max(k - 1, l - 1) = n$, and we conclude from $A(n)$ to be true that $k - 1 = l - 1$. Adding 1 to both sides yields $k = l$. This shows that $A(n + 1)$ is a true statement.

Can you find what is wrong here?

Question 5 *(taken from M.V. Day: "An Introduction to Proofs and the Mathematical Vernacular")* Suppose that x, y, z are positive real numbers. Prove that $x > z$ and $y^2 = xz$ together imply that $x > y > z$.

Question 6 Show that $2^{3^n} + 1$ is divisible by 3^{n+1} for every integer $n \geq 0$.
Hint: You may need the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Question 7 *(taken from P.J. Eccles: "An Introduction to Mathematical Reasoning: Numbers, Sets and Functions")* Let a_n be recursively defined by $a_1 = 1$ and $a_{n+1} = \frac{6a_n + 5}{a_n + 2}$ for $n \in \mathbb{N}$. Show that $0 < a_n < 5$.

Question 8 Let $n \geq 2$ be an integer. Then the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

cannot be an integer.

(a) Try to prove this statement with the help of the following fact (which you may use without proof): For every integer $n \geq 2$ there is a prime number p satisfying $p \leq n < 2p$.

(b) **Difficult:** Try to prove this statement without the previously mentioned fact, using that there is an integer $k \geq 0$ such that $2^k \leq n < 2^{k+1}$.

For both proofs you may need the following fact, which you can use without proof: every integer $m \geq 2$ can be written in a unique way as a product of primes (uniqueness means here that two factorisations of m into primes differ only by the order of the factors, e.g., $10 = 2 \cdot 5 = 5 \cdot 2$).

Question 9 Let $k \in \mathbb{N}$. Consider a square grid consisting of side length 2^k (i.e., the square grid consists of 2^{2k} unit squares). Remove one unit square anywhere in the grid. Show that you can cover the remaining area without overlaps by the following pieces, built up by 3 unit squares: