Answers to Number Problems

Question 1 Let

$$N = a \cdot 100 + b \cdot 10 + c$$

with $a > b > c$. Then

$$M = c \cdot 100 + b \cdot 10 + a.$$

Subtracting M from N leads to

$$N - M = (a - c) \cdot 100 + (c - a),$$

but we need to take into account that $-9 \leq c - a < 0$. So we have to modify the decimal representation of $N - M$ in the following way

$$A = N - M = (a - c - 1) \cdot 100 + 9 \cdot 10 + (10 + c - a),$$

then we have $1 \leq 10 + c - a \leq 9$ and $0 \leq a - c - 1 \leq 8$. Then B is given by

$$B = (10 + c - a) \cdot 100 + 9 \cdot 10 + (a - c - 1),$$

and

$$A + B = 9 \cdot 100 + 18 \cdot 10 + 9 = 10 \cdot 100 + 8 \cdot 10 + 9 = 1089.$$

Question 2 You may ask the question which numbers are possible for the last piece of paper to carry. The answer is:

The last piece of paper can never carry an odd number. The only possible numbers are all the even numbers between 0 *and* 100*. For each of these even numbers, there is a procedure that the last piece ends up with this number.*

Here are arguments which lead to this conclusion.

(a) The number of papers in the hat with an odd number is always even. This this true at the beginning (50 odd numbers $1, 3, 5, \ldots, 99$) and at each stage of the process the number of pieces carrying an odd number remain the same (if the two numbers drawn are not both odd) or drops by two (if the two numbers drawn are both odd). If there is only one piece of paper left in the hat, the pieces of paper carrying an odd number must be zero. This means the last piece of paper must carry an even number.

(b) It is obvious from the procedure that the papers in the hat can only carry numbers between 0 and 100.

(c) Here is a procedure that the last piece of paper carries the number 0: In the first 50 draws you take out the pairs $(50, 100), (49, 99), \dots, (1, 51)$. After this procedure, there are 50 pieces of paper, all carrying the number 50. In the next 25 draws you always take out pairs $(50, 50)$, so that you end up with 25 pieces of paper, all carrying the number 0. This leads necessarily to the situation that the last piece of paper carries the number 0.

(d) Assume that we have a situation where the number 1 is in the hat and also four distinct numbers $a, a+1, b, b+1$, each with multiplicity one. We describe a procedure after which these four numbers $a, a+1, b, b+1$ disappear and all the other numbers in the hat remain the same with unchanged multiplicities: Draw $(1, a+1)$ and return a into the hat. Draw (a, a) and return 0 into the hat. Draw $(b, b+1)$ and return 1 into the hat. Draw $(0, 1)$ and return 1 into the hat. Now the numbers $a, a+1, b, b+1$ are no longer in the hat.

(e) Now, choose an arbitrary number $2k$ with $k \in \{1, 2, \dots, 50\}$. At the beginning, there is obviously an even number of papers with numbers above $2k$ and also an even number of papers with number between 2 and $2k-1$, inclusively. By carrying out repeatedly procedure (d) with distinct numbers $a, a+1, b, b+1$ different from 1 and $2k$, we can manage to end up with the four numbers $1, 2, 3, 2k$ in case $2k \geq 4$ or $1, 2, 3, 4$ in case $2k = 2$. In the case $2k \geq 4$, you draw $(1, 3)$, $(2, 2)$, $(0, 2k)$ and end up with $2k$. In the ase $2k = 2$, you draw $(1, 4)$, $(3, 3)$, $(0, 2)$ and end up with 2.

Question 3

(a) The prime number 5 has multiplicity 24 in 100! (the only numbers between 1 and 100, and divisible by 5 are $5, 10, 15, 20, \dots, 100$, that is 20 numbers; of these, only $25, 50, 75, 100$ are divisible by 5^2, and none of these is divisible by 5^3). Similarly, one derives that the multiplicity of 2 in 100! is much higher, namely $97 = 50 + 25 + 12 + 6 + 3 + 1$. The largest power k such that 100! is divisible by k, is therefore 24. This is the number of zero digits at the end.

(b) There are manifold approaches to find good exponents. We start with a lower estimate on 9!:

$$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 = 720 \cdot 7 \cdot 8 \cdot 9 \geq 5000 \cdot 8 \cdot 9 = 40000 \cdot 9 = 3.6 \cdot 10^5.$$

This leads to the lower estimates

$$\begin{aligned} 10 \cdot 20 \cdots 90 \cdot 100 &\geq 3.6 \cdot 10^{16}, \\ 11 \cdot 21 \cdots 91 &\geq 3.6 \cdot 10^{14}, \\ &\vdots \\ 19 \cdot 29 \cdots 99 &\geq 3.6 \cdot 10^{14}. \end{aligned}$$

Putting everything together and using $3.6^2 \geq 10$, we obtain

$$100! \geq 9! \cdot 3.6^{10} \cdot 10^{16+9\cdot 14} \geq 3.6^{11} \cdot 10^{5+16+9\cdot 14} \geq 10^{5+5+16+9\cdot 14} = 10^{152}.$$

Next, we derive an upper estimate of 10!:

$$6! \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 720 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \leq 5100 \cdot 720 \leq 3.7 \cdot 10^6 \leq 4 \cdot 10^6.$$

Moreover, we have

$$1 \cdot 11 \cdot 21 \cdots 91 \leq 2 \cdot 12 \cdot 22 \cdots 92 \leq 10 \cdot 20 \cdot 30 \cdots 100 = (10!) \cdot 10^{10}.$$

This leads to

$$\begin{aligned} 100! \leq ((10!) \cdot 10^{10})^{10} = (10!)^{10} 10^{100} \leq (4 \cdot 10^6)^{10} 10^{100} = 2^{20} 10^{160} \\ = (1024)^2 10^{160} \leq 11^2 10^{164} \leq 10^{167}. \end{aligned}$$

In fact, we have

$$\begin{aligned} 100! = 93326215, 4439441526, 8169923885, 6266700490, 7159682643, \\ 8162146859, 2963895217, 5999932299, 1560894146, 3976156518, \\ 2862536979, 2082722375, 8251185210, 9168640000, 0000000000, \\ 0000000000 \approx 9.33 \cdot 10^{157}. \end{aligned}$$

(c) We have $\log(100!) = \sum_{j=1}^{100} \log(j)$. Since log is monotone increasing, we can estimate the sum from below and above by the integrals

$$\int_1^{100} \log(x)dx \leq \sum_{j=1}^{100} \log(j) \leq \int_1^{101} \log(x)dx.$$

Since $\int \log(x)dx = x\log(x) - x$, we obtain

$$361.52 \approx 100\log(100) - 99 \leq \log(100!) \leq 101\log(101) - 100 \approx 366.13.$$