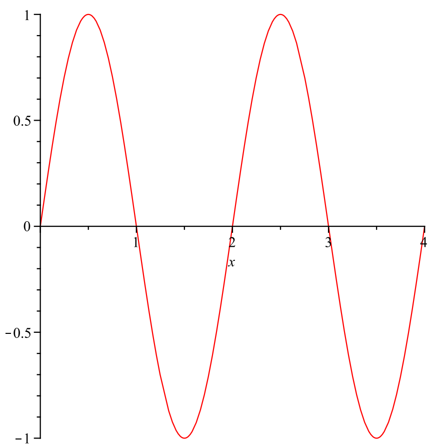
Answers to Preimages and Equivalence Relations Problems

Question 1 We have

$$\begin{aligned} f^{-1}(\{-1\}) &= \emptyset, \\ f^{-1}(\{0\}) &= \{(0,0,0)\}, \\ f^{-1}(\{1\}) &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}, \\ f^{-1}([1,2]) &= \{(x,y,z) \in \mathbb{R}^3 \mid 1 \leq x^2+y^2+z^2 \leq 2\}. \end{aligned}$$

This means that $f^{-1}(\{-1\})$ is the empty set, $f^{-1}(\{0\})$ is the set containing just the origin, and $f^{-1}(\{1\})$ is the Euclidean sphere around the origin of radius 1. Finally, $f^{-1}([1,2])$ is a closed Euclidean annulus, centered at the origin, with inner radius 1 and outer radius 2.

Question 2 The graph of $f(x)$ looks as follows:



We have

$$\begin{aligned} f^{-1}([0,1)) = \{x \in [0,4] \mid 0 \leq \sin(\pi x) < 1\} \\ = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \cup \left[2, \frac{5}{2}\right) \cup \left(\frac{5}{2}, 3\right] \cup \{4\}. \end{aligned}$$

Question 3

(a) Let $x \in f^{-1}(Y_1 \cap Y_2)$. This is equivalent to $f(x) \in Y_1 \cap Y_2$, which is equivalent to "$f(x) \in Y_1$ and $f(x) \in Y_2$". This, in turn, is equivalent to "$x \in f^{-1}(Y_1)$ and $x \in f^{-1}(Y_2)$, which is equivalent to "$x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Here, we proved in one go that every element of one set is also an element of the other set, and vice versa.

(b) There are manifold choices to establish this, for example $X_1 := [-2, 0]$ and $X_2 := [0, 2]$. Then

$$f(X_1 \cap X_2) = f(\{0\}) = \{0\}$$

and

$$f(X_1) \cap f(X_2) = f([-2, 0]) \cap f([0, 2]) = [0, 4] \cap [0, 4] = [0, 4],$$

that is $f(X_1 \cap x_2) \neq f(X_1) \cap f(X_2)$.

Question 4

(a) Let $x, x' \in X$. Assume that $g \circ f(x) = g \circ f(x')$, i.e. $g(f(x)) = g(f(x'))$. Since g is injective, this implies that $f(x) = f(x')$. Since f is injective, this implies that $x = x'$. This shows that $g \circ f$ is injective.

(b) Let $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $z = g(y)$. Since f is surjective, there exists $x \in X$ such that $y = f(x)$. This shows that $g \circ f$ is surjective.

(c) Let f, g be both bijective. Then (a) and (b) imply that $g \circ f : X \to Z$ is also bijective. In order to show $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, we need to show that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = x$$

for all $x \in X$. Since composition of functions is associative, we have

$$(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1} \circ (g^{-1} \circ g) \circ f(x) = \\ f^{-1} \circ \mathrm{id}_Y \circ f(x) = f^{-1} \circ f(x) = \mathrm{id}_X(x) = x,$$

where $\mathrm{id}_X : X \to X$ and $\mathrm{id}_Y : Y \to Y$ denote the identities on X and Y, respectively.

Question 5

(a) Transitivity is violated, since $1 \sim 0$ and $0 \sim 2$, but $1 \not\sim 2$.

(b) Reflexivity, Symmetry and Transitivity translate into $0 \in \mathbb{Q}$, $a \in \mathbb{Q} \Rightarrow -a \in \mathbb{Q}$ and $a, b \in \mathbb{Q} \Rightarrow a + b \in \mathbb{Q}$. We explain this in the case of Transitivity: If $x \sim y$ and $y \sim z$ we have $x - y, y - z \in \mathbb{Q}$. This implies $x - z = (x - y) + (y - z) \in \mathbb{Q}$, i.e., $x \sim z$.

(c) Note that $(x, y) \sim (x', y')$ if $x^2 - y^2 = (x')^2 - (y')^2$. Therefore, the equivalence classes of this equivalence relation are the preimages of $f(x, y) = x^2 - y^2$.

(d) Reflexivity is violated, since a nonzero vector $(x, y) \neq (0, 0)$ is obviously not orthogonal to itself.

(e) Transitivity is violated. Let $v_0 = 0 \in \mathbb{R}^n$. Then we have for any two vectors $v, w \in \mathbb{R}^n$: $v_0 \sim v$ and $v_0 \sim w$. Transitivity would lead to $v \sim w$, but there are obviously two non-zero vectors in $\mathbb{R}^n$, $n \geq 2$, which are not linearly dependent.

(f) Reflexivity is satisfied with $X = \mathrm{ID}_n$. We conclude Symmetry from the fact that $A = XBX^{-1}$ implies $B = X^{-1}A(X^{-1})^{-1}$. Finally, let us check Transitivity: Assume that $A \sim B$ and $B \sim C$. Then we can find invertible matrices X, Y such that $A = XBX^{-1}$ and $B = YCY^{-1}$. Then, we have

$$A = XBX^{-1} = XYCY^{-1}X^{-1} = (XY)C(XY)^{-1},$$

i.e., $A \sim C$, since XY is then also invertible.

(g) Reflexivity follows that a trivial bijection is given by the identity map $\mathrm{Id}_X : X \to X$. Symmetry follows from the fact that if $f : X \to Y$ is bijective, then $f^{-1} : Y \to X$ is also bijective. Finally, let us check Transitivity: Assume that $X \sim Y$ and $Y \sim Z$. Then there exist bijective maps $f : X \to Y$ and $g : Y \to Z$. By Question 4(c), we know that then $g \circ f : X \to Z$ is also bijective, i.e., $X \sim Z$.

(h) We obviously have $\int_0^1 f(x) - f(x)dx = 0$, so $f \sim f$, confirming Reflexivity. Now let $f \sim g$. Then we also have

$$0 = -\int_0^1 f(x) - g(x)dx = \int_0^1 g(x) - f(x)dx,$$

i.e., $g \sim f$, proving Symmetry. Now, assume that $f \sim g$ and $g \sim h$, i.e.,

$$0 = \int_0^1 f(x) - g(x) dx = \int_0^1 g(x) - h(x) dx.$$

This implies that

$$\int_0^1 f(x) - h(x) dx = \int_0^1 f(x) - g(x) dx + \int_0^1 g(x) - h(x) dx = 0 + 0 = 0,$$

i.e., $f \sim h$, proving Transitivity.

Question 6 We have Reflexivity $(a, b) \sim (a, b)$ because of $ab = ab$. Symmetry: Let $(a, b) \sim (c, d)$, i.e., $ad = bc$. Then we also have $cb = da$, i.e. $(c, d) \sim (a, b)$. Finally, let us check Transitivity: Assume that $(a, b) \sim (c, d)$, i.e., $ad = bc$ and $(c, d) \sim (e, f)$, i.e., $cf = de$. This implies that $adf = bcf = bde$ and, since $d \in \mathbb{N}$, $af = be$, i.e., $(a, b) \sim (e, f)$. Another way to check transitivity is to see that $(a, b) \sim (c, d)$ if $\frac{a}{b} = \frac{c}{d}$. So $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ translate into $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, which obviously implies $\frac{a}{b} = \frac{e}{f}$, i.e., $(a, b) \sim (e, f)$. To check that

$$[a, b] \otimes [c, d] := [ac, ad - bc] \tag{1}$$

is a well-defined operation means to check that this definition does not depend of the representatives of the equivalence relations. Assume that $[a, b] = [a', b']$ and $[c, d] = [c', d']$, i.e., $ab' = a'b$ and $cd' = c'd$. We just need to show that that then $[ac, ad - bc] = [a'c', a'd' - b'c']$, i.e., $ac(a'd' - b'c') = a'c'(ad - bc)$. This follows from

$$ac(a'd' - b'c') = aa'cd' - ab'cc' = aa'c'd - a'bcc' = a'c'(ad - bc).$$

Another way to see the well-definedness is to observe that if we identify $[a, b]$ with $\frac{b}{a}$, then $[a, b] \otimes [c, d]$ translates into $-\frac{b}{a} + \frac{d}{c} = \frac{ad-bc}{ac}$, which is well defined and independent of the representation of the involved rational numbers.

Question 7 We have $p(x) \sim p(x)$ since the trivial polynomial is divisible by $x^2 + 1$ (Reflexivity). Symmetry follows from the obvious fact that if $p(x) - q(x)$ is divisible by $x^2 + 1$, then so is $q(x) - p(x)$. Finally, if $p(x) \sim q(x)$ and $q(x) \sim r(x)$, then $p(x) - q(x) = a(x)(x^2 + 1)$ and $q(x) - r(x) = b(x)(x^2 + 1)$, which implies

$$p(x) - r(x) = (p(x) - q(x)) + (q(x) - r(x)) = (a(x) - b(x))(x^2 + 1),$$

i.e., $p(x) \sim r(x)$. This shows Transitivity.

(a) We have

$$(x^2+7)(x-3) = x^3 - 3x^2 + 7x - 21 = (x^2+1)(x-3) + 6x - 18.$$

This shows taht $(x^2+7)(x-3) - (6x-18)$ is divisible by x^2+1, i.e., $(x^2+7)(x-3) \sim 6x-18$, i.e., $[(x^2+7)(x-3)] = [6x-18]$.

(b) Assume that we have $p(x) = a_n x^n + \cdots + a_1 x + a_0$ with $n \geq 2$ and $a_n \neq 0$. Then we can write

$$p(x) = p_0(x) =$$
$$(x^2+1)a_n x^{n-2} + a_{n-1}x^{n-1} + (a_{n-2}-1)x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1 x + a_0.$$

This shows that $[p_0(x)] = [p_1(x)]$, setting

$$p_1(x) = a_{n-1}x^{n-1} + (a_{n-2}-1)x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1 x + a_0,$$

and $p_1(x)$ has a strictly lower degree than $p_0(x)$. We can continue with this reduction process until we end up with a polynomial $p_k(x) = b_1 x + b_0$ of degree at most one such that $[p_0(x)] = [p_k(x)]$. This shows that every equivalence class $[p(x)]$ has a representative of the form $b_1 x + b_2$ with $b_1, b_2 \in \mathbb{R}$.

(c) Note that a non-trivial linear real polynomial $ax+b$ is not divisible by the quadratic polynomial x^2+1. This shows that $[ax+b] \neq [a'x+b']$ if $(a,b) \neq (a',b')$. This fact, together with (b), imply that the map

$$[ax+b] \mapsto ai + b$$

is a bijection between the equivalence classes of $\mathbb{R}[x]$ and the set of complex numbers $\mathbb{C}$. Moreover, we have

$$(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd = ac(x^2+1) + (ad+bc)x + (bd-ac),$$

i.e.,

$$[(ax+b)(cx+d)] = [(ad+bc)x + (bd-ac)] \mapsto$$
$$(ad+bc)i + (bd-ac) = (ai+b)(ci+d).$$