Potpourri of Problems (Week 10)

Question 1 There is a capital logical mistake in the following arguments. Find and name this mistake: Let $u_1 > a > 0$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{a^2}{u_n})$. Assuming $u_n > a$ for some $n \in \mathbb{N}$, we want to show that also

$$u_{n+1} > a. \tag{1}$$

We have

$$\begin{array}{rrcl} & u_{n+1} &>& a, \\ \Rightarrow & u_n + \dfrac{a^2}{u_n} &>& 2a, \\ \Rightarrow & u_n^2 - 2au_n + a^2 &>& 0, \qquad \text{(multiplication by } u_n > 0\text{)}, \\ \Rightarrow & (u_n - a)^2 &>& 0. \end{array}$$

But since we assumed $u_n > a$, we also have $(u_n - a)^2 > 0$. Therefore, the above derivation shows that (1) is true.

Question 2 Which of the following equations is not correct:

$$\begin{aligned} \{\} &= \{x^2 \mid x \le -1 \text{ and } x > 2\} \\ &= \{x^2 \mid x \le -1\} \cap \{x^2 \mid x > 2\} \\ &= [1, \infty) \cap (4, \infty) \\ &= (4, \infty). \end{aligned}$$

Question 3 The following statements are all true. You do **not** need to prove them. Instead, formulate the corresponding contrapositive statements which are, therefore, also true:

(a) If the side lengths a, b, c of a triangle satisfy $a^2 + b^2 = c^2$ then the triangle is right–angled.

(b) If a sequence (x_n) of real numbers is monotone increasing and bounded from above then (x_n) is convergent.

(c) If a sequence $(x_n)_{n=1}^{\infty}$ of real numbers has a limit $x_\infty \in \mathbb{R}$ then every subsequence $(x_{n_i})_{i=1}^{\infty}$ converges also to the limit x_∞.

(d) If four points in the plane lie on a common circle then the opposite angles of the corresponding quadrilateral add up to 180^0.

Question 4 Insert in each of the following statements "necessary" or "sufficient" so that they become true statements. Decide for each statement whether it is still true if you insert "necessary and sufficient":

(a) Let A and B two finite sets. For a map $f : A \to B$ to be bijective it is ________ that $|A| = |B|$.

(b) Let (x_n) be a sequence of non-negative real numbers. For $((-1)^n x_n)$ to be convergent it is ________ that $x_n \to 0$.

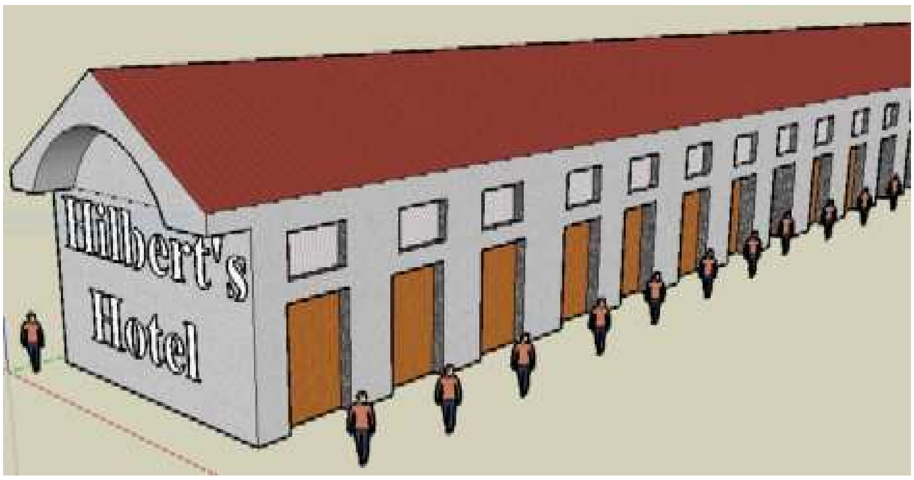
(c) For a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous it is ________ that f is differentiable.

(d) Let (a_n) be a sequence of real numbers. For a series $\sum_{n=1}^{\infty} a_n$ to be convergent it is ________ that $a_n \to 0$.

(e) For two vectors $v_1, v_2 \in \mathbb{R}^2$ to be linearly independent it is ________ that they are non-zero and orthogonal to each other.

(f) Let A, B be two 2×2 real matrices and Id the 2×2 identity matrix. For $AB = BA$ it is ________ that at least one the two matrices is a multiple of Id.

Question 5 Question 2 (Hilbert's Hotel) Hilbert's Hotel is an imaginary hotel with infinitely countably many single rooms numbered $1, 2, 3, \ldots$. Assume you are the receptionist of this hotel and that all rooms are already occupied with guests. Try to accomodate all new guests in each of the following situations.[1]



[1]The pictures are taken from
http://www.mathcs.org/analysis/reals/infinity/answers/hilbert-hotel.html and
http://www.tiptoptens.com/2011/11/19/top-10-best-christmas-gifts-for-fathers/

(a) Five new guests arrive and want to check in.

(b) A bus with infinitely countably many passengers arrives and all of them must be accomodated.

(c) The next day, all guests check out and the hotel is again empty, giving countably many personnel the opportunity to clean all the rooms. But a few hours later infinitely countably many buses arrive, each of them carrying infinitely countably many passengers.

Question 6 (Challenging) Decide for each of the following sets whether they are necessarily countable or not:

(a) A set of disjoint open discs in $\mathbb{R}^2$ (the discs may have different positive radii). Note that we do not ask for the cardinality of points in these discs but of the discs themselves as elements of the set.

(b) The set of all convergent sequences $(x_n)_{n=1}^{\infty}$ whose elements x_n are natural numbers.

(c) The set of all sequences $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = 0$, and whose elements are rational numbers.

MERRY CHRISTMAS AND HAPPY NEW YEAR 2013!!!