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# Riemannian Geometry IV

**Problems, set 11.**

**Exercise 26.** As in the lecture, let $G$ be a Lie group, $H \subset G$ be a closed subgroup, $\pi : G \to G/H$ be the canonical projection, $\langle \cdot, \cdot \rangle_e$ be an $Ad(H)$-invariant inner product on $T_eG$, $V \subset T_eG$ be the orthogonal complement to $T_eH \subset T_eG$ with respect to $\langle \cdot, \cdot \rangle_e$, and $\Phi$ the restriction of $D\pi(e) : T_eG \to T_{eH}G/H$ to the subspace $V$. Prove the following statements:

(a) $T_eH = \ker D\pi(e)$. (You may use without proof that $D\pi(e) : T_eG \to T_{eH}G/H$ is surjective.)

(b) $\Phi : V \to T_{eH}G/H$ is an isomorphism.

(c) $V$ is $Ad(H)$-invariant. (**Hint:** The fact that $Ad(h_1)Ad(h_2) = Ad(h_1h_2)$ might be useful.)

**Exercise 27.** In this exercise, we introduce a left-invariant Riemannian metric on the homogeneous space $SL(2, \mathbb{R})/SO(2)$. Let $G = SL(2, \mathbb{R})$ and $H = SO(2)$.

(a) Show that $T_eH = \{\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\}$.

(b) For $A, B \in T_eG = \{C \in M(2, \mathbb{R}) \mid \mathrm{tr}(C) = 0\}$ (where $\mathrm{tr}(C)$ denotes the trace of the matrix $C$), define

$$\langle A, B \rangle_e := 2\mathrm{tr}(AB^\top).$$

Check that $\langle \cdot, \cdot \rangle_e$ is symmetric and $Ad(H)$-invariant.

(c) Let $V = \langle\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}\} \subset T_eG$. Show that $V$ is the orthogonal complement of $T_eH$ with respect to $\langle \cdot, \cdot \rangle_e$.

(d) Let $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \in V$. Check that $A, B$ are an orthonormal basis of $V$ with respect to $\langle \cdot, \cdot \rangle_e$.

Recall that we obtain the Riemannian metric on $SL(2, \mathbb{R})/SO(2)$ via left-translation of $\langle \cdot, \cdot \rangle_e$ (as in the lectures). Recall also (see Example 17) that $SL(2, \mathbb{R})/SO(2)$ is diffeomorphic tp the hyperbolic upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ via

$$SL(2, \mathbb{R})/SO(2) \to \mathrm{H}^2, \quad A \cdot SO(2) \mapsto f_A(i),$$

where $f_A(z) = \frac{az+b}{cz+d}$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(e) Calculate the tangent vectors $v, w \in T_i \mathrm{H}^2$ corresponding to $A, B$ in part (d) of this exercise.

This exercise shows that the Riemannian metric, constructed on $SL(2, \mathbb{R})/SO(2)$, coincides with the hyperbolic metric on $\mathbb{H}^2$, given by

$$\langle v, w \rangle_z = \frac{\langle v, w \rangle_0}{\mathrm{Im}(z)}.$$