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Riemannian Geometry IV

Problems, set 12 (to be handed in on 31 January 2011 in the afternoon lecture).

Exercise 28. *First Variation Formula of energy*: Let $F : (-\epsilon, \epsilon) \times [a, b] \to M$ be a variation of a differentiable curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$ and X be its variational vector field. Let $E : (-\epsilon, \epsilon)$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2} \int_a^b \| \frac{\partial F}{\partial t}(s, t) \|^2 dt.$$

Show that

$$E'(0) = \langle X(b), c'(b) \rangle - \langle X(a), c'(a) \rangle - \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle dt.$$

Discuss the particular cases when

(i) c is a geodesic,

(ii) F is a proper variation,

(iii) c is a geodesic and F is a proper variation.

Let $c : [a, b] \to M$ be a curve connecting p and q (not necessarily parametrised proportional to arc length). Show the following statements:

(iv) $E'(0) = 0$ for every proper variation implies that c is a geodesic.

(v) Assume that c minimises the energy amongst all curves $\gamma : [a, b] \to M$ which connect p and q. Then c is a geodesic.