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Riemannian Geometry IV

Problems, set 17.

Do **Exercise 43** as homework for this week. The cumulative homework over the previous weeks will be collected on **7 March 2011** in the afternoon lecture.

Exercise 42. Let (M, g) be a connected Riemannian manifold of dimension $n \geq 3$ with the following property: There is a function $f : M \to \mathbb{R}$ such that, for every $p \in M$, the sectional curvature of **all** 2-planes $\Sigma \subset T_pM$ satisfies

$$K(\Sigma) = f(p).$$

The goal of this exercise is to show that then f is a constant function, i.e., there is a $C \in \mathbb{R}$ such that $f(p) = C$ for all $p \in M$ (*Schur's Theorem*).

Hint: Define $R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$ and

$$R'(X, Y, Z, W) = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle.$$

Use Exercises 39 and 41 to show that $\nabla R(X, Y, Z, W, U) = (Uf)R'(X, Y, Z, W)$ (for the definition of the covariant derivative of a tensor, see Exercise 19). Use the **Second Bianchi Identity** (see Exercise 38) to show that

$$\begin{aligned}(Tf)(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle)& \\ + (Zf)(\langle X, T\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, T\rangle)& \\ + (Wf)(\langle X, Z\rangle\langle Y, T\rangle - \langle X, T\rangle\langle Y, Z\rangle) &= 0.\end{aligned}$$

Fix a point $p \in M$ and choose $X(p), Z(p) \in T_PM$ arbitrary. Because $n \geq 3$, we can choose W, Y such that

$$\langle Z(p), W(p)\rangle_p = \langle Z(p), Y(p)\rangle_p = \langle Y(p), W(p)\rangle_p = 0,$$

and $\|Y(p)\| = 1$. Choose $T = Y$. Show that this choice yields

$$\langle (Wf)(p)Z(p) - (Zf)(p)W(p), X(p)\rangle(p) = 0,$$

and conclude that we have $(Zf)(p) = 0$. Since $p \in M$ and $Z(p) \in T_pM$ was arbitrary, this implies that f is constant on connected components of M. The statement then follows from the connectedness of M.

Exercise 43. Prove the Second Variation Formula of Length: Let $F : (-\epsilon, \epsilon) \times [a, b] \to M$ be a **proper** variation of a **geodesic** $c : [a, b] \to M$ with $\|c'\| = 1$. Let $X(t) = \frac{\partial F}{\partial s}(0, t)$ be its variational vector field and

$$X^\perp(t) = X(t) - \langle X(t), c'(t) \rangle c'(t)$$

be the component of X orthogonal to $c'(t)$. Let $l : (-\epsilon, \epsilon) \to [0, \infty)$ be the associated length functional. Then we have

$$l''(0) = \int_a^b \left(\left\| \frac{DX^\perp}{dt} \right\|^2 - K(\mathrm{span}\{c', X^\perp\}) \|X^\perp\|^2 \right) dt,$$

where $K(\mathrm{span}\{c', X^\perp\})$ denotes the sectional curvature of the plane spanned by c' and $X^\perp$. If $X^\perp(t) = 0$, i.e., if $c'(t)$ and $X^\perp(t)$ are linearly dependent, we set $K(\mathrm{span}\{c'(t), X^\perp(t)\}) = 0$.

Hint: Study carefully the proof of the first variation formula for length and at the second variation formula for energy (both presented in the lectures), and adapt these methods to the current situation.