Riemannian Geometry IV

Problems, set 2 (to be handed in on 25 October 2010 in the afternoon lecture).

Exercise 4. Let M be a differentiable manifold, $U \subset M$ open and $\varphi = (x_1, \dots, x_n) : U \to V_1 \subset \mathbb{R}^n$, $\psi = (y_1, \dots, y_n) : U \to V_2 \subset \mathbb{R}^n$ be two coordinate charts. Show for $p \in U$:

$$\frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^{n} \frac{\partial(y_j \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot \frac{\partial}{\partial y_j}\Big|_p,$$

where $y_j \circ \varphi^{-1} : V_1 \to \mathbb{R}$ and $\frac{\partial(y_j \circ \varphi^{-1})}{\partial x_i}$ is the classical partial derivative in the coordinate direction x_i of $\mathbb{R}^n$. **Hint:** Blow $f \circ \varphi^{-1}$ up to the expression $f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$, and apply the chain rule.

Exercise 5. This exercise is useful to become familiar with the notions introduced in the lectures.

Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ be the standard two-dimensional sphere and $\mathbb{R}P^2$ be the real projective plane and $\pi : S^2 \to \mathbb{R}P^2$ be the canonical projection $p \mapsto p/\sim$. Let

$$c : (-\epsilon, \epsilon) \to S^2, \quad c(t) = (\cos t \cos(2t), \cos t \sin(2t), \sin t)$$

and

$$f : \mathbb{R}P^2 \to \mathbb{R}, \quad f(\mathbb{R}(z_1, z_2, z_3)^\top) = \frac{(z_1 + z_2 + z_3)^2}{z_1^2 + z_2^2 + z_3^2}.$$

(a) Let $\gamma = \pi \circ c$. Calculate $\gamma'(0)(f)$.

(b) Let (φ, U) be the following coordinate chart of $\mathbb{R}P^2$: $U = \{\mathbb{R}(z_1, z_2, z_3)^\top \mid z_1 \neq 0\} \subset \mathbb{R}P^2$ and

$$\varphi : U \to \mathbb{R}^2, \quad \varphi(\mathbb{R}(z_1, z_2, z_3)^\top) = \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}\right).$$

Let $\varphi = (x_1, x_2)$. Express $\gamma'(t)$ in the form

$$\alpha_1(t)\frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \alpha_2(t)\frac{\partial}{\partial x_2}\Big|_{\gamma(t)}.$$