Dr. Norbert Peyerimhoff, Durham University 25/10/2010

Riemannian Geometry IV

Problems, set 3.

Exercise 6. Prove that the tangent space of the Lie group $SO(n) \subset M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ at the identity $e \in SO(n)$ is given by

$$T_e SO(n) = \{A \in M(n,\mathbb{R}) \mid A^\top = -A\},$$

i.e., the space of all skew-symmetric $n \times n$-matrices. **Hint:** You may use that we have, componentwise, $(AB)'(t) = A'(t)B(t) + A(t)B'(t)$, for the product of any two matrix-valued curves.

Exercise 7. Let G be a Lie group of dimension n and $H \subset G$ be a closed subgroup of dimension k. The quotient space G/H is the set of left-cosets

$$G/H = \{gH \mid g \in G\},$$

where $gH := \{gh \mid h \in H\}$. Let $\pi : G \to G/H$ be the canonical projection $\pi(g) = gH$. The aim of this (challenging) exercise is to introduce coordinate charts of G/H and to show that the coordinate changes are differentiable. Assume there is an open neighbourhood U of the identity element $e \in G$ and a diffeomorphism $\varphi : U \to V_1 \times V_2 \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ with the following properties:

(a) $\varphi(e) = 0 \in V_1 \times V_2 \subset \mathbb{R}^n$,

(b) $\varphi^{-1}(\{x\} \times V_2) = g_x H \cap U$, where $g_x = \varphi^{-1}(\{x\} \times \{0\}) \in G$,

(c) if $x_1, x_2 \in V_1$, $x_1 \neq x_2$, then $g_{x_1} H \neq g_{x_2} H$.

The construction of this pair (U, φ) is non-trivial and requires deeper results. You can take it for granted. Draw a picture to illustrate the properties of φ.

(a) Use π and φ to construct a coordinate chart in a neighbourhood of $eH \in G/H$.

(b) Use the group action to construct coordinate charts in a neighborhood of $gH \in G/H$ for every $g \in G$. (You may use the maps $L_g, R_g : G \to G$, $L_g(g') = gg'$ and $R_g(g') = g'g$, the left and right multiplications by g.)

(c) Proof that coordinate changes of overlapping coordinate charts are differentiable. Note that the maps $L_g, R_g : G \to G$, are differentiable by the definition of a Lie group.