Dr. Norbert Peyerimhoff, Durham University 13/12/2010

Riemannian Geometry IV

Solutions, set 10.

Exercise 23. (a) Let $A = \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix}$. We have

$$A^2 = \begin{pmatrix} 0 & 0 & t^2 & 0 \\ 0 & 0 & 0 & t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & 0 & t^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^k = 0 \qquad \text{for all } k \geq 4.$$

So the power series $\mathrm{Exp}(A)$ terminates after 4 terms and we conclude that

$$\mathrm{Exp}(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 = \begin{pmatrix} 1 & t & t^2/2 & t^3/(3!) \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Let $B = tcI$, where I denotes the 4×4 identity matrix and let A be as in (a). Then we have $\mathrm{Exp}(B) = e^{tc}I$ and A and B commute. This implies that

$$\mathrm{Exp}\left(t \begin{pmatrix} c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{pmatrix} \right) = \mathrm{Exp}(A+B)$$

$$= \mathrm{Exp}(B)\mathrm{Exp}(A) = e^{tc} \begin{pmatrix} 1 & t & t^2/2 & t^3/(3!) \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 24. (a) We have

$$\langle\langle Ad(h^{-1})v_1, Ad(h^{-1})v_2\rangle\rangle_e = \int_G \langle Ad(g^{-1})Ad(h^{-1})v_1, Ad(g^{-1})Ad(h^{-1})v_2\rangle_e \, dvol(g)$$
$$= \int_G \langle Ad((hg)^{-1})v_1, Ad((hg)^{-1})v_2\rangle_e \, dvol(g).$$

Let $f : G \to \mathbb{R}$, $f(g) = \langle Ad(g^{-1})v_1, Ad(g^{-1})v_2\rangle_e$. Then we have

$$\langle\langle Ad(h^{-1})v_1, Ad(h^{-1})v_2\rangle\rangle_e = \int_G f(hg)\, dvol(g) = \int_G f(g)\, dvol(g)$$
$$= \int_G \langle Ad(g^{-1})v_1, Ad(g^{-1})v_2\rangle_e \, dvol(g) = \langle\langle v_1, v_2\rangle\rangle_e.$$

(b) We have

$$\langle\langle DR_h(e)v_1, DR_h(e)v_2\rangle\rangle_h = \langle DL_{h^{-1}}(h)DR_h(e)v_1, DL_{h^{-1}}(h)DR_h(e)v_2\rangle\rangle_e$$
$$= \langle Ad(h^{-1})v_1, Ad(h^{-1})v_2\rangle\rangle_e = \langle\langle v_1, v_2\rangle\rangle_e.$$

Exercise 25. The relation in the hint implies that

$$\langle X, \nabla_Y Y\rangle =$$
$$\frac{1}{2}\left(Y\langle X, Y\rangle + Y\langle X, Y\rangle - X\langle Y, Y\rangle + \langle Y, [X,Y]\rangle + \langle Y, [X,Y]\rangle - \langle X, [Y,Y]\rangle\right) =$$
$$\frac{1}{2}\left(\langle Y, [X,Y]\rangle + \langle Y, [X,Y]\rangle\right),$$

since the inner product of two left invariant vector fields is constant and, therefore, the first three derivatives of the right hand side of the relation vanish. Moreover, we have $[Y,Y] = 0$. So we conclude that

$$\langle X, \nabla_Y Y\rangle = \langle Y, [X,Y]\rangle.$$

The bi-invariance implies that

$$\langle [Y,X], Y\rangle = -\langle Y, [Y,X]\rangle = -\langle [Y,X], Y\rangle,$$

so $\langle [Y,X], Y\rangle = 0$. This gives us $\langle X, \nabla_Y Y\rangle = 0$ for all left invariant X, so we have $\nabla_Y Y = 0$ for all left invariant Y. Using this fact, we calculate

$$0 = \nabla_{X+Y} X+Y = \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y = \nabla_X Y + \nabla_Y X = 2\nabla_X Y - [X,Y].$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X,Y].$$