Dr. Norbert Peyerimhoff, Durham University 21/2/2011

Riemannian Geometry IV

Solutions, set 16.

Exercise 39. (a) (i)-(iv) are straighforward calculations.

(b) Since $K(\Sigma) = C$ for all 2-dimensional subspaces $\Sigma \subset T_pM$, we have for any choice of two vectors v, w:

$$\langle R(v,w)w, v\rangle = C\left(\|v\|^2\|w\|^2 - \langle v,w\rangle^2\right).$$

This holds obviously true for linear independent vectors v, w, and in the case of linear dependent vectors v, w one easily checks that both expressions of the equation are equal to zero. So we can use the identity

$$(v,w,w,v) = (v,w,w,v)'$$

for general pairs of vectors v, w. We obtain on the one side, using linearity of $(\cdot,\cdot,\cdot,\cdot)'$:

$$\begin{aligned}(v_1, v_2+v_3, v_2+v_3, v_1) &= (v_1, v_2+v_3, v_2+v_3, v_1)' = \\ &= (v_1,v_2,v_2,v_1)' + (v_1,v_2,v_3,v_1)' + (v_1,v_3,v_2,v_1)' + (v_1,v_3,v_3,v_1)',\end{aligned}$$

and on the other side

$$\begin{aligned}&(v_1, v_2+v_3, v_2+v_3, v_1) = \\ &= (v_1,v_2,v_2,v_1) + (v_1,v_2,v_3,v_1) + (v_1,v_3,v_2,v_1) + (v_1,v_3,v_3,v_1) = \\ &= (v_1,v_2,v_2,v_1)' + (v_1,v_2,v_3,v_1)' + (v_1,v_3,v_2,v_1)' + (v_1,v_3,v_3,v_1)',\end{aligned}$$

which leads to

$$(v_1,v_2,v_3,v_1) + (v_1,v_3,v_2,v_1) = (v_1,v_2,v_3,v_1)' + (v_1,v_3,v_2,v_1)'. \qquad (1)$$

By the symmetries, we obtain yield

$$(v_1,v_3,v_2,v_1) = (v_2,v_1,v_1,v_3) = (v_1,v_2,v_3,v_1),$$

and the same holds for $(\cdot,\cdot,\cdot,\cdot)'$, so (1) simplifies to

$$2(v_1, v_2, v_3, v_1) = 2(v_1, v_2, v_3, v_1)',$$

finishing (b).

(c) Using (b), we obtain on the one side

$$\begin{aligned}&(v_1+v_4, v_2, v_3, v_1+v_4) = (v_1+v_4, v_2, v_3, v_1+v_4)' = \\ &\qquad = (v_1, v_2, v_3, v_1)' + (v_1, v_2, v_3, v_4)' + (v_4, v_2, v_3, v_1)' + (v_4, v_2, v_3, v_4)',\end{aligned}$$

and on the other side

$$\begin{aligned}&(v_1+v_4, v_2, v_3, v_1+v_4) = \\ &\quad = (v_1, v_2, v_3, v_1) + (v_1, v_2, v_3, v_4) + (v_4, v_2, v_3, v_1) + (v_4, v_2, v_3, v_4) = \\ &\qquad = (v_1, v_2, v_3, v_1)' + (v_1, v_2, v_3, v_4)' + (v_4, v_2, v_3, v_1)' + (v_4, v_2, v_3, v_4)'.\end{aligned}$$

Comparing both expressions, we conclude that

$$(v_1, v_2, v_3, v_4) + (v_4, v_2, v_3, v_1) = (v_1, v_2, v_3, v_4)' + (v_4, v_2, v_3, v_1)',$$

finishing (c).

(d) We obtain directly from (c):

$$(v_1, v_2, v_3, v_4) - (v_1, v_2, v_3, v_4)' = -(v_4, v_2, v_3, v_1) + (v_4, v_2, v_3, v_1)'.$$

Using the symmetries, we derive

$$-(v_4, v_2, v_3, v_1) = -(v_3, v_1, v_4, v_2) = (v_3, v_1, v_2, v_4),$$

and the same identity for $(\cdot,\cdot,\cdot,\cdot)'$, so we end up with

$$(v_1, v_2, v_3, v_4) - (v_1, v_2, v_3, v_4)' = (v_3, v_1, v_2, v_4) - (v_3, v_1, v_2, v_4)'.$$

(e) Using (d), Bianchi's first identity and property (ii), we conclude that

$$\begin{aligned}&3\left((v_1, v_2, v_3, v_4) - (v_1, v_2, v_3, v_4)'\right) = \left((v_1, v_2, v_3, v_4) - (v_1, v_2, v_3, v_4)'\right) + \\ &\quad + \left((v_3, v_1, v_2, v_4) - (v_3, v_1, v_2, v_4)'\right) + \left((v_2, v_3, v_1, v_4) - (v_2, v_3, v_1, v_4)'\right) = \\ &\qquad = \left((v_1, v_2, v_3, v_4) + (v_3, v_1, v_2, v_4) + (v_2, v_3, v_1, v_4)\right) - \\ &\qquad - \left((v_1, v_2, v_3, v_4)' + (v_3, v_1, v_2, v_4)' + (v_2, v_3, v_1, v_4)'\right) = 0 - 0 = 0,\end{aligned}$$

proving the statement of (e). Replacing $(\cdot,\cdot,\cdot,\cdot)$ and $(\cdot,\cdot,\cdot,\cdot)'$ by their original meanings, yields identity (1) of Exercise 39.

Exercise 40. Homework! Solution will be provided later!

Exercise 41. (a) We have

$$\begin{aligned}
&\nabla T(X_1, X_2, X_3, X_4, Y) \\
&\qquad = Y(T_1(X_1,X_2)T_2(X_3,X_4)) - \sum_{i=1}^{4} T(X_1,\dots,\nabla_Y X_i,\dots,X_4) \\
&\qquad = T_1(X_1,X_2)\underbrace{(Y(T_2(X_3,X_4)) - T_2(\nabla_Y X_3) - T_2(\nabla_Y X_4))}_{=\nabla T_2(X_3,X_4,Y)=0} + \\
&\qquad\qquad T_2(X_3,X_4)\underbrace{(Y(T_1(X_1,X_2)) - T_1(\nabla_Y X_1) - T_1(\nabla_Y X_2))}_{=\nabla T_1(X_1,X_2,Y)=0} = 0.
\end{aligned}$$

(b) Let $T(X,Y) = \langle X,Y\rangle$. Since ∇ is Riemannian, we have

$$\nabla T(X,Y,Z) = Z(\langle X,Y\rangle) - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle = 0.$$

Note that $R'(X,Y,Z,W) = T(X,W)T(Y,Z) - T(X,Z)T(Y,W)$. (a) implies then that we have $\nabla R' = 0$.

(c) If (M,g) is a manifold with constant sectional curvature $C \in \mathbb{R}$, we have by Exercise 39:

$$\begin{aligned}
R(X,Y,Z,W) &= \langle R(X,Y)Z, W\rangle \\
&= C(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) = CR'(X,Y,Z,W).
\end{aligned}$$

Then $\nabla R = C\nabla R' = 0$ follows from (b).