Riemannian Geometry IV

Solutions, set 18.

Exercise 44.

(a) The tangent space of S_r^2 at $p \in S_r^2$ is given by

$$T_p S_r^2 = p^\perp.$$

Now, using the results of Example 22, we obtain

$$\frac{D}{dt} X(t) = \left(\frac{d}{dt}(0, \cos t, 0) \right)^\perp = ((0, -\sin t, 0))^\perp,$$

where $v^\perp$ is taken at $c(t) = (r \cos t, 0, r \sin t)$. Since $(0, -\sin t, 0) \perp c(t)$, we conclude that

$$\frac{D}{dt} X(t) = (0, -\sin t, 0).$$

Similarly, we conclude that

$$\frac{D^2}{dt^2} X(t) = (0, -\cos t, 0) = -X(t).$$

Now, using the notation of Exercise 37, we have

$$\begin{aligned} \left.\frac{\partial}{\partial x_1}\right|_{c(t)} &= (-r \sin t, 0, r \cos t) = c'(t), \\ \left.\frac{\partial}{\partial x_2}\right|_{c(t)} &= (0, r \cos t, 0) = rX(t). \end{aligned}$$

Using the results of Exercise 37(a), we conclude that

$$\begin{aligned} R(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}) \frac{\partial}{\partial x_1} &= \nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} - \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1} \\ &= \nabla_{\frac{\partial}{\partial x_2}}(0) - \nabla_{\frac{\partial}{\partial x_1}} \left(-\tan x_1 \frac{\partial}{\partial x_2} \right) \\ &= (1 + \tan^2 x_1) \frac{\partial}{\partial x_2} - \tan^2 x_1 \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2}. \end{aligned}$$

This implies that

$$R(X(t),c'(t))c'(t) = r\left(R(\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1})\frac{\partial}{\partial x_1}\right)(c(t)) = r\frac{\partial}{\partial x_2}\Big|_{c(t)} = rX(t).$$

Bringing everything together, we conclude that

$$\frac{D^2}{dt^2}X(t) + R(X(t),c'(t))c'(t) = -X(t) + X(t) = 0,$$

i.e., X satisfies the Jacobi equation.

Exercise 45.

(a) We conclude from Proposition 6.4 that

$$R(v_1,v_2)v_3 = K(\langle v_2,v_3\rangle v_1 - \langle v_1,v_3\rangle v_2).$$

This implies

$$R(J,c')c' = K(\langle c',c'\rangle J - \langle J,c'\rangle c').$$

Since $\|c'\|^2 = 1$ and $J \perp c'$, we obtain

$$R(J,c')c' = KJ.$$

(b) We only consider the case $K > 0$, all other cases are similar. The vector field $J(t) = \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}}Z_2(t)$ satisfies $J(0) = Z_1(0)$ and

$$\frac{DJ}{dt}(t) = -\sqrt{K}\sin(t\sqrt{K})Z_1(t) + \cos(t\sqrt{K})Z_2(t),$$

which implies $\frac{DJ}{dt}(0) = Z_2(0)$. Obviously, we have

$$\frac{D^2J}{dt^2}(t) = -K\cos(t\sqrt{K})Z_1(t) - \sqrt{K}\sin(t\sqrt{K})Z_2(t) = -KJ(t),$$

and therefore we obtain

$$\frac{D^2J}{dt^2}(t) + KJ(t) = 0,$$

i.e., J satisfies the Jacobi equation.

Exercise 46.

(a) We have

$$f'(t) = \frac{d}{dt}\Big|_{t=0} \langle J(t), J(t) \rangle = 2\langle \frac{D}{dt} J(t), J(t) \rangle$$

and

$$f''(t) = 2\left(\langle \frac{D^2}{dt^2} J(t), J(t) \rangle + \left\| \frac{D}{dt} J(t) \right\|^2 \right).$$

Using Jacobi's equation, we conclude

$$f''(t) = 2\left(-\langle R(J(t), c'(t))c'(t), J(t) \rangle + \left\| \frac{D}{dt} J(t) \right\|^2 \right).$$

We have $\langle R(J(t), c'(t))c'(t), J(t) \rangle = 0$ if $J(t), c'(t)$ are linear dependent and, otherwise, for $\sigma = \mathrm{span}(J(t), c'(t)) \subset T_{c(t)}M$,

$$\langle R(J(t), c'(t))c'(t), J(t) \rangle = K(\sigma) \left(\|J(t)\|^2 \|c'(t)\|^2 - (\langle J(t), c'(t) \rangle)^2 \right) \leq 0,$$

since sectional curvature is non-positive. This shows that $f''(t)$, as a sum of two non-negative terms, is greater or equal to zero.

(b) If there were a conjugate point $q = c(t_2)$ to a point $p = c(t_1)$ along the geodesic c, then we would have a non-vanishing Jacobi field J along c with $J(t_1) = 0$ and $J(t_2) = 0$. This would imply that the convex, non-negative function $f(t) = \|J(t)\|^2$ would have zeroes at $t = t_1$ and $= t_2$. This would force f to vanish identically on the interval $[t_1, t_2]$, which would imply that J vanishes as well, which is a contradiction.

Exercise 47.

(a) We know that $\nabla R = 0$. Let $\frac{D}{dt}$ denote covariant derivative along c. Then we have, for parallel vector fields X, Y, Z along c that

$$\begin{aligned} 0 = \nabla R(X, Y, Z, c')(t) &= \frac{D}{dt} R(X(t), Y(t))Z(t) \\ &- R(\underbrace{\frac{D}{dt} X(t)}_{=0}, Y(t))Z(t) - R(X(t), \underbrace{\frac{D}{dt} Y(t)}_{=0})Z(t) - R(X(t), Y(t)) \underbrace{\frac{D}{dt} Z(t)}_{=0} \\ &= \frac{D}{dt} R(X(t), Y(t))Z(t). \end{aligned}$$

This shows that $R(X, Y)Z$ is parallel.

(b) The symmetries of R yield

$$\begin{aligned}\langle K_v(w_1), w_\rangle = \langle R(w_1,v)v, w_2\rangle = \langle R(v,w_2)w_1, v\rangle = -\langle R(w_2,v)w_1,v\rangle \\ = \langle R(w_2,v)v, w_1\rangle = \langle K_v(w_2), w_1\rangle.\end{aligned}$$

(c) Since K_v is symmetric, we can find an orthonormal basis $w_1, \ldots, w_n \in T_pM$ with $K_v(w_i) = \lambda_i w_i$. We know, by (a), that $K_{c'(t)}(W_i(t)) = R(W_i(t), c'(t))W_i(t)$ is parallel and, since $K_{c'(0)}(W_i(0)) = K_v(w_i) = \lambda_i w_i$, we must have

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t),$$

since parallel vector fields V along c are uniquely determined by their initial values $V(0) \in T_pM$.

(d) Let J be a Jacobi field along c. Then J satisfies the Jacobi equation

$$\frac{D^2 J}{dt^2} J + R(J, c')c' = 0.$$

Since $W_1, \ldots, W_n$ are a parallel on-basis along c, we obtain, by taking inner product with W_i:

$$\begin{aligned}&\left\langle \frac{D^2 J}{dt^2} J, W_i \right\rangle + \langle R(J,c')c', W_i\rangle \\ &\qquad = \frac{d^2}{dt^2} \sum_j J_j \langle W_j, W_i\rangle + \sum_j J_j \langle R(W_j, c')c', W_i\rangle \\ &\qquad\qquad = J_i'' + \sum_j J_j \lambda_j \langle W_j, W_i\rangle = J_i'' + \lambda_i J_i.\end{aligned}$$

(e) The unique solution of $J_i''(t) + \lambda_i J_i(t) = 0$, $J_i(0) = 0$ (up to scalar multiples) is given by

$$J_i(t) = \begin{cases} \sin(t\sqrt{\lambda_i}) & \text{if } \lambda_i > 0, \\ t & \text{if } \lambda_i = 0, \\ \sinh(t\sqrt{-\lambda_i}) & \text{if } \lambda_i < 0. \end{cases}$$

So J_i has zeroes for positive t only if $\lambda_i > 0$, and these are precisely at $t = \pi k/\sqrt{\lambda_i}$. The corresponding Jacobi fields with $J(0) = 0$ and $\frac{DJ}{dt}(0) = w_i$ produce the conjugate points $c(\pi k/\sqrt{\lambda_i})$.