Shimizu's lemma for quaternionic hyperbolic space

Wensheng Cao
School of Mathematics and Computational Science,
Wuyi University, Jiangmen,
Guangdong 529020, P.R. China
e-mail: wenscao@yaliyun.com

John R. Parker
Department of Mathematical Sciences,
Durham University, Durham DH1 3LE, England
j.r.parker@durham.ac.uk

Abstract

We prove a version of Shimizu's lemma for quaternionic hyperbolic space. Namely, consider groups of quaternionic hyperbolic isometries containing a parabolic map fixing infinity. We show that any element of such a group not fixing infinity has an isometric sphere whose radius is bounded by a function of the parabolic translation length at its centre.

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1 Introduction

1.1 The context

The *hyperbolic spaces* (that is rank 1 symmetric spaces of non-compact type) are $\mathbf{H}^n_{\mathbb{F}}$ where $\mathbb{F}$ is one of the real numbers, the complex numbers, the quaternions or the octonions (and in the last case $n = 2$), see Chen and Greenberg [5]. A map in $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{F}})$ is *parabolic* if it has a unique fixed point and this point lies on $\partial\mathbf{H}^n_{\mathbb{F}}$. Parabolic isometries of $\mathbf{H}^2_{\mathbb{R}}$ and $\mathbf{H}^3_{\mathbb{R}}$, that is parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$, are particularly simple: they are (conjugate to) Euclidean translations. In all the other cases, there are more complicated parabolic maps, which are conjugate to Euclidean screw motions.

Shimizu's lemma [22] gives a necessary condition for a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ containing a parabolic element to be discrete. If one normalises so that the parabolic fixed point is ∞, then Shimizu's lemma says that the isometric spheres of any group element not fixing infinity have bounded radius, the bound being the Euclidean translation length. Equivalently, it says that the horoball with height the Euclidean translation length is precisely invariant (that is elements of the group either map the horoball to itself or to a disjoint horoball). Shimizu's lemma was generalised to $\mathrm{PSL}(2,\mathbb{C})$ by Leutbecher [17] and to subgroups of $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ containing a translation by Wielenberg [24]. Ohtake gave examples showing that, for $n \geq 4$, subgroups of $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ containing a more general parabolic map can have isometric spheres of arbitrarily large radius, or equivalently there can be no precisely invariant horoball [19]. Finally Waterman [23] gave a version of Shimizu's lemma for more general parabolic maps, by showing that each isometric sphere is bounded by a function of the parabolic translation length at its centre. Recently Erlandsson and Zakeri [6, 7] have constructed precisely invariant regions contained in a horoball with better asymptotics than those of Waterman.

It is then natural to ask for versions of Shimizu's lemma associated to other rank 1 symmetric spaces. The holomorphic isometry groups of $\mathbf{H}^n_{\mathbb{C}}$ and $\mathbf{H}^n_{\mathbb{H}}$ are $\mathrm{PU}(n,1)$ and $\mathrm{PSp}(n,1)$ respectively. Kamiya generalised Shimizu's lemma to subgroups of $\mathrm{PU}(n,1)$ or $\mathrm{PSp}(n,1)$ containing a vertical Heisenberg translation [13]. For subgroups of $\mathrm{PU}(n,1)$ containing a general Heisenberg translation, Parker [20, 21] gave version of Shimizu's lemma both in terms of a bound on the radius of isometric spheres and a precisely invariant horoball or sub-horospherical region. This was generalised to $\mathrm{PSp}(n,1)$ by Kim and Parker [16]. Versions of Shimizu's lemma for subgroup of $\mathrm{PU}(2,1)$ containing a screw parabolic map were given by Jiang, Kamiya and Parker [10, 14]. Kim claimed the main result of [10] holds for $\mathrm{PSp}(2,1)$ [15]. But in fact, he failed to consider all possible types of screw parabolic map (in the language below, he assumed $\mu = 1$). Our result completes the project began by Kamiya [13] by giving a full version of Shimizu's lemma for any parabolic isometry of $\mathbf{H}^n_{\mathbb{C}}$ or $\mathbf{H}^n_{\mathbb{H}}$ for all $n \geq 2$.

Shimizu's lemma is a special case of Jørgensen's inequality [12], which is among the most important results about real hyperbolic 3-manifolds. Jørgensen's inequality has also been generalised to other hyperbolic spaces. Versions for isometry groups of $\mathbf{H}^2_{\mathbb{C}}$ containing a loxodromic or elliptic map were given by Basmajian and Miner [1] and Jiang, Kamiya and Parker [9]. These results were extended to $\mathbf{H}^2_{\mathbb{H}}$ by Kim and Parker [16] and Kim [15]. Cao and Parker [3, 4] obtained generalised Jørgensen's inequalities in $\mathbf{H}^n_{\mathbb{H}}$ for groups containing a loxodromic or elliptic map. Finally, Markham and Parker [18] obtained a version of Jørgensen's inequality for the isometry groups of $\mathbf{H}^2_{\mathbb{O}}$ with certain types of loxodromic map.

1.2 Statements of the main results

The purpose of this paper is to obtain a generalised version of Shimizu's lemma for parabolic isometries of quaternionic hyperbolic n-space, and in particular for screw parabolic isometries. In order to state our main results, we need to use some notation and facts about quaternions and quaternionic hyperbolic n-space.

We will show in Section 2.3 that a general parabolic isometry of quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$ can be normalised to the form

$$T = \begin{pmatrix} \mu & -\sqrt{2}\tau^*\mu & (-\|\tau\|^2 + t)\mu \\ 0 & U & \sqrt{2}\tau\mu \\ 0 & 0 & \mu \end{pmatrix}, \tag{1}$$

where $\tau \in \mathbb{H}^{n-1}$, t is a purely imaginary quaternion, $U \in \mathrm{Sp}(n-1)$ and μ is a unit quaternion satisfying

$$\begin{cases} U\tau = \mu\tau,\ U^*\tau = \overline{\mu}\tau,\ \mu\tau \neq \tau\overline{\mu} & \text{if } \tau \neq 0, \\ \mu t \neq t\overline{\mu} & \text{if } \tau = 0 \text{ and } \mu \neq \pm 1, \\ t \neq 0 & \text{if } \tau = 0 \text{ and } \mu = \pm 1. \end{cases} \tag{2}$$

We call a parabolic element of form (1) a *Heisenberg translation* if $U = I_{n-1}$ and $\mu = 1$, and we say that it is *screw parabolic* otherwise. We remark that even for $n = 2$ it is possible to find screw parabolic maps with $\mu \neq \pm 1$ and $\tau \neq 0$. This is the point overlooked by Kim in [15].

If μ is a unit quaternion and $\zeta \in \mathbb{H}^{n-1}$, the map $\zeta \longmapsto \mu\zeta\overline{\mu}$ is linear. For U and μ as above, consider the following linear maps:

$$B_{U,\mu} : \zeta \longmapsto U\zeta - \zeta\mu, \qquad B_\mu : \zeta \longmapsto \mu\zeta - \zeta\mu.$$

Define $N_{U,\mu}$ and N_μ to be their spectral norms, that is

$$N_{U,\mu} = \max\Big\{ \|B_{U,\mu}\zeta\| \ : \ \zeta \in \mathbb{H}^{n-1} \text{ and } \|\zeta\| = 1 \Big\}, \tag{3}$$

$$N_\mu = \max\Big\{ \|B_\mu\zeta\| \ : \ \zeta \in \mathbb{H}^{n-1} \text{ and } \|\zeta\| = 1 \Big\} = 2\big|\mathrm{Im}(\mu)\big|. \tag{4}$$

Note that $U^*\zeta - \zeta\overline{\mu} = U^*\zeta\mu\overline{\mu} - U^*U\zeta\overline{\mu} = -U^*(U\zeta - \zeta\mu)\overline{\mu}$. Therefore $N_{U^*,\overline{\mu}} = N_{U,\mu}$. We remark that $N_\mu = 0$ if and only if $\mu = \pm 1$, and $N_{U,\mu} = 0$ if and only if $\mu = \pm 1$ and $U = \mu I_{n-1}$.

We may identify the boundary of $\mathbf{H}^n_{\mathbb{H}}$ with the $4n-1$ dimensional generalised Heisenberg group, which is $\mathfrak{N}_{4n-1} = \mathbb{H}^{n-1} \times \mathrm{Im}\mathbb{H}$ with the group law

$$(\zeta_1, v_1) \cdot (\zeta_2, v_2) = \big(\zeta_1 + \zeta_2, v_1 + v_2 + 2\mathrm{Im}(\zeta_2^*\zeta_1)\big).$$

There is a natural metric called, the *Cygan metric*, on $\mathfrak{N}_{4n-1}$. Any parabolic map T is a Cygan isometry of $\mathfrak{N}_{4n-1}$. The natural projection from $\mathfrak{N}_{4n-1}$ to $\mathbb{H}^{n-1}$ given by $\Pi : (\zeta, v) \longmapsto \zeta$ is called *vertical projection*. The vertical projection of T is a Euclidean isometry of $\mathbb{H}^{n-1}$.

An element S of $\mathrm{Sp}(n,1)$ not fixing ∞ is clearly not a Cygan isometry. However there is Cygan sphere with centre $S^{-1}(\infty)$, called the *isometric sphere of* S, that is sent by S to the Cygan sphere of the same radius, centred at $S(\infty)$. We call this radius $r_S = r_{S^{-1}}$. Our first main result is the following theorem relating the radius of the isometric spheres of S and S^{-1}, the Cygan translation length of T at their centres and the Euclidean translation length of the vertical projection of T at the vertical projections of the centres.

Theorem 1.1 *Let Γ be a discrete subgroup of $\mathrm{PSp}(n,1)$ containing the parabolic map T given by (1). Suppose that the quantities $N_{U,\mu}$ and N_μ defined by (3) and (4) satisfy $N_\mu < 1/4$ and $N_{U,\mu} < (3 - 2\sqrt{2 + N_\mu})/2$. Define*

$$K = \frac{1}{2}\Big(1 + 2N_{U,\mu} + \sqrt{1 - 12N_{U,\mu} + 4N_{U,\mu}^2 - 4N_\mu}\Big). \tag{5}$$

If S is any other element of Γ not fixing ∞ and with isometric sphere of radius r_S then

$$r_S^2 \leq \frac{\ell_T\big(S^{-1}(\infty)\big)\ell_T\big(S(\infty)\big)}{K} + \frac{4\big\|\Pi T S^{-1}(\infty) - \Pi S^{-1}(\infty)\big\| \, \big\|\Pi T S(\infty) - \Pi S(\infty)\big\|}{K(K - 2N_{U,\mu})}. \tag{6}$$

If $\mu = 1$ then Theorem 1.1 becomes simpler and it also applies to subgroups of $\mathrm{PU}(n,1)$:

Corollary 1.2 *Let Γ be a discrete subgroup of $\mathrm{PU}(n,1)$ or $\mathrm{PSp}(n,1)$ containing the parabolic map T given by (1) with $\mu = 1$. Suppose $N_U = N_{U,1}$ defined by (3) satisfies $N_U < (\sqrt{2} - 1)^2/2$. Define*

$$K = \frac{1}{2}\Big(1 + 2N_U + \sqrt{1 - 12N_U + 4N_U^2}\Big).$$

If S is any other element of Γ not fixing ∞ and with isometric sphere of radius r_S then

$$r_S^2 \leq \frac{\ell_T\big(S^{-1}(\infty)\big)\ell_T\big(S(\infty)\big)}{K} + \frac{4\big\|\Pi T S^{-1}(\infty) - \Pi S^{-1}(\infty)\big\| \, \big\|\Pi T S(\infty) - \Pi S(\infty)\big\|}{K(K - 2N_U)}.$$

We remark that if T is a Heisenberg translation $T_{\tau,t}$ then $N_{U,\mu} = N_\mu = 0$. Therefore $K = 1$. Moreover, $\big\|\Pi T S^{-1}(\infty) - \Pi S^{-1}(\infty)\big\| = \big\|\Pi T S(\infty) - \Pi S(\infty)\big\| = \|\tau\|$ and so Theorem 1.1, or Corollary 1.2, is just Theorem 4.8 of Kim-Parker [16]. If in addition $\tau = 0$ then $\ell_T(S^{-1}(\infty)) = \ell_T(S(\infty)) = |t|^{1/2}$ and we recover Theorem 3.2 of Kamiya [13].

For a parabolic map T of the form (1), consider the following sub-horospherical region:

$$\mathcal{U}_T = \left\{(\zeta, v, u) \in \mathbf{H}^n_{\mathbb{H}} \ : \ u > \frac{\ell_T(z)^2}{K - N_\mu} + \frac{4(2K - N_\mu)\|\Pi T(z) - \Pi(z)\|^2}{(K - N_\mu)\big((K - N_\mu)(K - 2N_{U,\mu}) - 2N_{U,\mu}K\big)}\right\}. \quad (7)$$

Also, using the definitions of $N_{U,\mu}$, N_μ and K one may check

$$(K - N_\mu)(K - 2N_{U,\mu}) - 2N_{U,\mu}K = (K - 4N_{U,\mu})2N_{U,\mu} + K(K - 2N_{U,\mu})^2,$$

which is positive since $K - 4N_{U,\mu} > (1 - 6N_{U,\mu})/2 > 0$. Note that when $\mu = \pm 1$, including the case of $\mathrm{PU}(n, 1)$, then we have the much simpler formula, generalising equation (3.1) of [21]:

$$\mathcal{U}_T = \left\{(\zeta, v, u) \in \mathbf{H}^n_{\mathbb{H}} \ : \ u > \frac{\ell_T(z)^2}{K} + \frac{8\|\Pi T(z) - \Pi(z)\|^2}{K(K - 4N_{U,\mu})}\right\}.$$

If H is a subgroup of G then we say a set $\mathcal{U}$ is *precisely invariant under H in G* if $T(\mathcal{U}) = \mathcal{U}$ for all $T \in H$ and $S(\mathcal{U}) \cap \mathcal{U} = \emptyset$ for all $S \in G - H$. Our second main result is a restatement of Theorem 1.1 in terms of a precisely invariant sub-horospherical region.

Theorem 1.3 *Let G be a discrete subgroup of $\mathrm{PSp}(n, 1)$. Suppose that G_∞ the stabiliser of ∞ in G is a cyclic group generated by a parabolic map of the form* (1)*. Suppose that $N_{U,\mu}$ and N_μ defined by* (3) *and* (4) *satisfy $N_\mu < 1/4$ and $N_{U,\mu} < (3 - 2\sqrt{2 + N_\mu})/2$ and let K be given by* (5)*. Then the sub-horospherical region $\mathcal{U}_T$ given by* (7) *precisely invariant under G_∞ in G.*

1.3 Outline of the proofs

All proofs of Shimizu's lemma, and indeed of Jørgensen's inequality, follow the same general pattern, see [13, 10, 16]. One considers the sequence $S_{j+1} = S_j T S_j^{-1}$. From this sequence one constructs a dynamical system involving algebraic or geometrical quantities involving S_j. The aim is to give conditions under which S_0 is in a basin of attraction guaranteeing S_j tends to T as j tends to infinity.

The structure of the remaining sections of this paper is as follows. In Section 2, we give the necessary background material for quaternionic hyperbolic space. In Section 3 we prove that Theorem 1.3 follows from Theorem 1.1. In Section 4 we construct our dynamical system. This involves the radius of the isometric spheres of S_j and S_j^{-1} and the translations lengths of T and its vertical projection at their centres. We establish recurrence relations involving these quantities for S_{j+1} and the same quantities for S_j. This lays a foundation for our proof of Theorem 1.1 in Sections 5 and 6. In Section 5 we rewrite the condition (6) in terms of this dynamical system, (Theorem 5.1) and show that it means we are in a basin of attraction. Finally, in Section 6, we show this implies S_j converges to T as j tends to infinity.

2 Background

2.1 Quaternionic hyperbolic space

We give the necessary background material on quaternionic hyperbolic geometry in this section. Much of the background material can be found in [5, 8, 16].

We begin by recalling some basic facts about the quaternions $\mathbb{H}$. Elements of $\mathbb{H}$ have the form $z = z_1+z_2\mathbf{i}+z_3\mathbf{j}+z_4\mathbf{k} \in \mathbb{H}$ where $z_i \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $\overline{z} = z_1-z_2\mathbf{i}-z_3\mathbf{j}-z_4\mathbf{k}$ be the *conjugate* of z, and $|z| = \sqrt{\overline{z}z} = \sqrt{z_1^2+z_2^2+z_3^2+z_4^2}$ be the *modulus* of z. We define $\mathrm{Re}(z) = (z+\overline{z})/2$ to be the *real part* of z, and $\mathrm{Im}(z) = (z-\overline{z})/2$ to be the *imaginary part* of z. Two quaternions z and w are *similar* if there is a non-zero quaternion q so that $w = qzq^{-1}$. Equivalently, z and w have the same modulus and the same real part. Let $X = (x_{ij}) \in M_{p\times q}$ be a $p \times q$ matrix over $\mathbb{H}$. Define the Hilbert-Schmidt norm of X to be $\|X\| = \sqrt{\sum_{i,j}|x_{ij}|^2}$. Also the Hermitian transpose of X, denoted X^*, is the conjugate transpose of X in $M_{q\times p}$.

Let $\mathbb{H}^{n,1}$ be the quaternionic vector space of quaternionic dimension $n+1$ with the quaternionic Hermitian form

$$\langle \mathbf{z},\, \mathbf{w}\rangle = \mathbf{w}^*H\mathbf{z} = \overline{w}_1 z_{n+1} + \overline{w}_2 z_2 + \cdots + \overline{w}_n z_n + \overline{w}_{n+1} z_1, \tag{8}$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors in $\mathbb{H}^{n,1}$ with entries $z_1, \ldots, z_{n+1}$ and $w_1, \ldots, w_{n+1}$ respectively, and H is the Hermitian matrix

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Following Section 2 of [5], let

$$V_0 = \Big\{\mathbf{z} \in \mathbb{H}^{n,1} - \{0\} : \langle \mathbf{z},\, \mathbf{z}\rangle = 0\Big\}, \quad V_- = \Big\{\mathbf{z} \in \mathbb{H}^{n,1} : \langle \mathbf{z},\, \mathbf{z}\rangle < 0\Big\}.$$

We define an equivalence relation $\sim$ on $\mathbb{H}^{n,1}$ by $\mathbf{z} \sim \mathbf{w}$ if and only if there exists a non-zero quaternion λ so that $\mathbf{w} = \mathbf{z}\lambda$. Let $[\mathbf{z}]$ denote the equivalence class of $\mathbf{z}$. Let $\mathbb{P} : \mathbb{H}^{n,1}-\{0\} \longrightarrow \mathbb{HP}^n$ be the *right projection* map given by $\mathbb{P} : \mathbf{z} \longmapsto [\mathbf{z}]$. If $z_{n+1} \neq 0$ then $\mathbb{P}$ is given by

$$\mathbb{P}(z_1, \ldots, z_n, z_{n+1})^T = (z_1 z_{n+1}^{-1}, \cdots, z_n z_{n+1}^{-1})^T \in \mathbb{H}^n.$$

We also define

$$\mathbb{P}(z_1, 0, \ldots, 0, 0)^T = \infty.$$

The Siegel domain model of *quaternionic hyperbolic n-space* is defined to be $\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_-)$ with boundary $\partial\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_0)$. It is clear that $\infty \in \partial\mathbf{H}^n_{\mathbb{H}}$. The Bergman metric on $\mathbf{H}^n_{\mathbb{H}}$ is given by the distance formula

$$\cosh^2\frac{\rho(z,w)}{2} = \frac{\langle \mathbf{z},\, \mathbf{w}\rangle\langle \mathbf{w},\, \mathbf{z}\rangle}{\langle \mathbf{z},\, \mathbf{z}\rangle\langle \mathbf{w},\, \mathbf{w}\rangle}, \quad \text{where} \quad z, w \in \mathbf{H}^n_{\mathbb{H}}, \quad \mathbf{z} \in \mathbb{P}^{-1}(z), \mathbf{w} \in \mathbb{P}^{-1}(w).$$

This expression is independent of the choice of lifts $\mathbf{z}$ and $\mathbf{w}$.

Quaternionic hyperbolic space is foliated by horospheres based at a boundary point, which we take to be ∞. Each horosphere has the structure of the one point compactification of $4n-1$

dimensional Heisenberg group $\mathfrak{N}_{4n-1} \cup \{\infty\}$. We define *horospherical coordinates* on $\overline{\mathbf{H}_{\mathbb{H}}^n} - \{\infty\}$ as $z = (\zeta, v, u)$ where $u \in [0, \infty)$ is the height of the horosphere containing z and $(\zeta, v) \in \mathfrak{N}_{4n-1}$ is a point of this horosphere. If $u = 0$ then z is in $\partial\mathbf{H}_{\mathbb{H}}^n - \{\infty\}$ which we identify with $\mathfrak{N}_{4n-1}$ by writing $(\zeta, v, 0) = (\zeta, v)$. Where necessary, we lift points of $\overline{\mathbf{H}_{\mathbb{H}}^n}$ written in horospherical coordinates to $V_0 \cup V_-$ via the map $\psi : \big(\mathfrak{N}_{4n-1} \times [0, \infty)\big) \cup \{\infty\} \longrightarrow V_0 \cup V_-$ given by

$$\psi(\zeta, v, u) = \begin{pmatrix} -\|\zeta\|^2 - u + v \\ \sqrt{2}\zeta \\ 1 \end{pmatrix}, \quad \psi(\infty) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The *Cygan metric* on the Heisenberg group is the metric corresponding to the norm

$$\big|(\zeta, v)\big|_H = \big|\|\zeta\|^2 + v\big|^{1/2} = \big(\|\zeta\|^4 + |v|^2\big)^{1/4}.$$

It is given by

$$d_H\big((\zeta_1, v_1), (\zeta_2, v_2)\big) = \big|(\zeta_1, v_1)^{-1}(\zeta_2, v_2)\big|_H = \Big|\|\zeta_1 - \zeta_2\|^2 + v_1 - v_2 - 2\mathrm{Im}(\zeta_2^*\zeta_1)\Big|^{1/2}.$$

As on page 303 of [16], we extend the Cygan metric to $\overline{\mathbf{H}_{\mathbb{H}}^n} - \{\infty\}$ by

$$d_H\big((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)\big) = \Big|\|\zeta_1 - \zeta_2\|^2 + |u_1 - u_2| + v_1 - v_2 - 2\mathrm{Im}(\zeta_2^*\zeta_1)\Big|^{1/2}.$$

2.2 The group $\mathrm{Sp}(n, 1)$

The group $\mathrm{Sp}(n, 1)$ is the subgroup of $\mathrm{GL}(n+1, \mathbb{H})$ preserving the Hermitian form given by (8). That is, $S \in \mathrm{Sp}(n, 1)$ if and only if $\langle S(\mathbf{z}), S(\mathbf{w})\rangle = \langle \mathbf{z}, \mathbf{w}\rangle$ for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{H}^{n,1}$. From this we find $S^{-1} = H^{-1}S^*H$. That is S and S^{-1} have the form:

$$S = \begin{pmatrix} a & \gamma^* & b \\ \alpha & A & \beta \\ c & \delta^* & d \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} \overline{d} & \beta^* & \overline{b} \\ \delta & A^* & \gamma \\ \overline{c} & \alpha^* & \overline{a} \end{pmatrix}, \tag{9}$$

where $a, b, c, d \in \mathbb{H}$, A is an $(n-1) \times (n-1)$ matrix over $\mathbb{H}$, and $\alpha, \beta, \gamma, \delta$ are column vectors in $\mathbb{H}^{n-1}$.

Using the identities $I_{n+1} = SS^{-1}$ we see that the entries of S must satisfy:

$$1 = a\overline{d} + \gamma^*\delta + b\overline{c}, \tag{10}$$
$$0 = a\overline{b} + \|\gamma\|^2 + b\overline{a}, \tag{11}$$
$$0 = \alpha\overline{d} + A\delta + \beta\overline{c}, \tag{12}$$
$$I_{n-1} = \alpha\beta^* + AA^* + \beta\alpha^*, \tag{13}$$
$$0 = \alpha\overline{b} + A\gamma + \beta\overline{a}, \tag{14}$$
$$0 = c\overline{d} + \|\delta\|^2 + d\overline{c}. \tag{15}$$

Similarly, equating the entries of $I_{n+1} = S^{-1}S$ yields:

$$\begin{aligned} 1 &= \overline{d}a + \beta^*\alpha + \overline{b}c, \\ 0 &= \overline{d}\gamma^* + \beta^* A + \overline{b}\delta^*, \\ 0 &= \overline{d}b + \|\beta\|^2 + \overline{b}d, \\ 0 &= \delta a + A^*\alpha + \gamma c, \\ I_{n-1} &= \delta\gamma^* + A^*A + \gamma\delta^*, \\ 0 &= \overline{c}a + \|\alpha\|^2 + \overline{a}c. \end{aligned}$$

An $(n-1)\times(n-1)$ quaternionic matrix U is in $\mathrm{Sp}(n-1)$ if and only if $UU^* = U^*U = I_{n-1}$. Using the above equations, we can verify the following lemma.

Lemma 2.1 *(c.f.[16, Lemma 1.1]) If S is as above then $A - \alpha c^{-1}\delta^*$ and $A - \beta b^{-1}\gamma^*$ are in* $\mathrm{Sp}(n-1)$. *Also we have*

$$\begin{aligned} \beta - \alpha c^{-1}d &= -(A - \alpha c^{-1}\delta^*)\delta\overline{c}^{-1}, \\ \gamma - \delta\overline{c}^{-1}\overline{a} &= -(A - \alpha c^{-1}\delta^*)^*\alpha c^{-1}, \\ \alpha - \beta b^{-1}a &= -(A - \beta b^{-1}\gamma^*)\gamma\overline{b}^{-1}, \\ \delta - \gamma\overline{b}^{-1}\overline{d} &= -(A - \beta b^{-1}\gamma^*)^*\beta b^{-1}. \end{aligned}$$

It is obvious that V_0 and V_- are invariant under $\mathrm{Sp}(n,1)$. This means that if we can show that the action of $\mathrm{Sp}(n,1)$ is compatible with the projection $\mathbb{P}$ then we can make $\mathrm{Sp}(n,1)$ act on quaternionic hyperbolic space and its boundary. The action of $S \in \mathrm{Sp}(n,1)$ on $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ is given as follows. Let $\mathbf{z} \in V_- \cup V_0$ be a vector that projects to z. Then

$$S(z) = \mathbb{P}S\mathbf{z}.$$

Note that if $\widetilde{\mathbf{z}}$ is any other lift of z, then $\widetilde{\mathbf{z}} = \mathbf{z}\lambda$ for some non-zero quaternion λ. We have

$$\mathbb{P}S\widetilde{\mathbf{z}} = \mathbb{P}S\mathbf{z}\lambda = \mathbb{P}S\mathbf{z} = S(z)$$

and so this action is independent of the choice of lift. The key point here is that the group acts on the left and projection acts on the right, hence they commute.

Let S have the form (9). If $c = 0$ then from (15) we have $\|\delta\| = 0$ and so δ is the zero vector in $\mathbb{H}^{n-1}$. Similarly, α is also the zero vector. This means that S (projectively) fixes ∞. On the other hand, if $c \neq 0$ then S does not fix ∞. Moreover, $S^{-1}(\infty)$ and $S(\infty)$ in $\mathfrak{N}_{4n-1} = \partial\mathbf{H}^n_{\mathbb{H}} - \{\infty\}$ have Heisenberg coordinates

$$S^{-1}(\infty) = \big(\delta\overline{c}^{-1}/\sqrt{2},\, \mathrm{Im}(\overline{d}\overline{c}^{-1})\big), \quad S(\infty) = \big(\alpha c^{-1}/\sqrt{2},\, \mathrm{Im}(ac^{-1})\big).$$

For any $r > 0$, it is not hard to check (compare Lemma 3.4 of [21]) that S sends the Cygan sphere with centre $S^{-1}(\infty)$ and radius r to the Cygan sphere with centre $S(\infty)$ and radius $\widetilde{r} = 1/|c|r$. The *isometric sphere* of S is the Cygan sphere with radius $r_S = 1/|c|^{1/2}$ centred at $S^{-1}(\infty)$. It is sent by S to the isometric sphere of S^{-1}, which is the sphere with centre $S(\infty)$ and radius r_S. In particular, if r and $\widetilde{r}$ are as above, then $\widetilde{r} = r_S^2/r$.

We define $\mathrm{PSp}(n,1) = \mathrm{Sp}(n,1)/\{\pm I_{n+1}\}$, which is the group of holomorphic isometries of $\mathbf{H}^n_{\mathbb{H}}$. Following Chen and Greenberg [5], we say that a non-trivial element g of $\mathrm{Sp}(n,1)$ is:

(i) *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{H}}$;

(ii) *parabolic* if it has exactly one fixed point, and this point lies in $\partial\mathbf{H}^n_{\mathbb{H}}$;

(iii) *loxodromic* if it has exactly two fixed points, both lying in $\partial\mathbf{H}^n_{\mathbb{H}}$.

2.3 Parabolic elements of $\mathrm{Sp}(n,1)$

The main aim of this section is to show that any parabolic motion T can be normalised as form given by (1).

Lemma 2.2 *(Johnson [11]) Consider the affine map on $\mathbb{H}$ given by $T_0 : z \longmapsto \nu z\overline{\mu} + \tau$ where $\tau \in \mathbb{H} - \{0\}$ and μ, $\nu \in \mathbb{H}$ with $|\mu| = |\nu| = 1$.*

(i) If ν is not similar to μ then T_0 has a fixed point.

(ii) If $\nu = \mu$ then T_0 has a fixed point in $\mathbb{H}$ if and only if $\mu\tau = \tau\overline{\mu}$.

We now characterise parabolic elements of $\mathrm{Sp}(n,1)$ (compare Theorem 3.1(iii) of [2]).

Proposition 2.3 *Let $T \in \mathrm{Sp}(n,1)$ be a parabolic map that fixes ∞. Then T may be conjugated into the standard form* (1). *That is*

$$T = \begin{pmatrix} \mu & -\sqrt{2}\tau^*\mu & \big(-\|\tau\|^2 + t\big)\mu \\ 0 & U & \sqrt{2}\tau\mu \\ 0 & 0 & \mu \end{pmatrix},$$

where $(\tau, t) \in \mathfrak{N}_{4n-1}$, $U \in \mathrm{Sp}(n-1)$ and $\mu \in \mathbb{H}$ with $|\mu| = 1$ satisfying (2)*. That is*

$$\begin{cases} U\tau = \mu\tau,\ U^*\tau = \overline{\mu}\tau,\ \mu\tau \neq \tau\overline{\mu} & \text{if } \tau \neq 0, \\ \mu t \neq t\overline{\mu} & \text{if } \tau = 0 \text{ and } \mu \neq \pm 1, \\ t \neq 0 & \text{if } \tau = 0 \text{ and } \mu = \pm 1. \end{cases}$$

Recall that if $U = I_{n-1}$ and $\mu = 1$ (or $U = -I_{n-1}$ and $\mu = -1$) then T is a Heisenberg translation. Otherwise, we say that U is screw parabolic.

Proof: Suppose that T is written in the general form (9). Then if T fixes ∞, it must be block upper triangular. That is $c = 0$ and $\alpha = \delta = 0$, the zero vector in $\mathbb{H}^{n-1}$. It easily follows that $a\overline{d} = 1$. If T is non-loxodromic, then $|a| = |d| = 1$ and so $a = d$. We define $\mu := a = d \in \mathbb{H}$ with $|\mu| = 1$.

If $o = (0,0)$ is the origin in $\mathfrak{N}_{4n-1}$, then suppose T maps o to $(\tau, t) \in \mathfrak{N}_{4n-1}$. This means that

$$bd^{-1} = -\|\tau\|^2 + t, \quad \beta d^{-1} = \sqrt{2}\tau.$$

Hence $b = \big(-\|\tau\|^2 + t\big)\mu$ and $\beta = \sqrt{2}\tau\mu$. Also, $A \in \mathrm{Sp}(n-1)$ and so we write $A = U$. It is easy to see from (14) that $U\gamma + \sqrt{2}\tau = 0$. Hence, T has the form

$$T = \begin{pmatrix} \mu & -\sqrt{2}\tau^*U & \big(-\|\tau\|^2 + t\big)\mu \\ 0 & U & \sqrt{2}\tau\mu \\ 0 & 0 & \mu \end{pmatrix}.$$

Since T fixes ∞ and is assumed to be parabolic, we need to find conditions on U, μ and τ that imply T does not fix any finite point of $\mathfrak{N}_{4n-1} = \partial\mathbf{H}^n_{\mathbb{H}} - \{\infty\}$.

Without loss of generality, we may suppose that U is a diagonal map whose entries u_i all satisfy $|u_i| = 1$. Writing the entries of ζ and $\tau \in \mathbb{H}^{n-1}$ as ζ_i and τ_i for $i = 1, \ldots, n-1$, we see that a fixed point (ζ, v) of T is a simultaneous solution to the equations

$$\begin{aligned} -\|\zeta\|^2 + v &= \mu\big(-\|\zeta\|^2 + v\big)\overline{\mu} - 2\tau^* U\zeta\overline{\mu} - \|\tau\|^2 + t, \\ \zeta_i &= u_i\zeta_i\overline{\mu} + \tau_i, \end{aligned}$$

for $i = 1, \ldots, n-1$. If any of the equations $\zeta_i = u_i\zeta_i\overline{\mu} + \tau_i$ has a solution, then conjugating by a translation if necessary, we assume this solution is 0.

If all the equations $\zeta_i = u_i\zeta_i\overline{\mu} + \tau_i$ have a solution, then, as above, $\zeta = 0$ and so $\tau = 0$. The first equation becomes

$$v = \mu v\overline{\mu} + t.$$

By Johnson's theorem, if $\mu \neq \pm 1$ this has no solution provided $\mu t \neq t\overline{\mu}$. Clearly, if $\mu = \pm 1$ then it has no solution if and only if $t \neq 0$.

On the other hand, if there are some values of i for which $\zeta_i = u_i\zeta_i\overline{\mu} + \tau_i$ has no solution, then by Johnson's theorem, for each such value of i, the corresponding u_i must be similar to μ (and $\tau_i \neq 0$ else 0 is a solution). Hence, without loss of generality, we may choose coordinates so that whenever $\tau_i \neq 0$ we have $u_i = \mu$. In particular, $u_i\tau_i = \mu\tau_i$ and so $U\tau = \mu\tau$. Furthermore, again using Johnson's theorem, $\mu\tau \neq \tau\overline{\mu}$.

Observe that $u_i\tau_i = \mu\tau_i$ implies

$$\overline{u}_i\tau_i = \overline{u}_i(\mu\tau_i)(\tau_i^{-1}\overline{\mu}\tau_i) = \overline{u}_i(u_i\tau_i)(\tau_i^{-1}\overline{\mu}\tau_i) = \overline{\mu}\tau_i.$$

Hence $U^*\tau = \overline{\mu}\tau$, or equivalently $\tau^* U = \tau^*\mu$ and so T has the required form. □

Note that if $U\tau = \mu\tau = \tau\overline{\mu}$ then $\zeta = \tau(1 - \overline{\mu}^2)^{-1}$ is a fixed point of $\zeta \longmapsto U\zeta\overline{\mu} + \tau$. Hence in this case, we may take $\tau = 0$. Furthermore, if $\tau = 0$ and $\mu t = t\overline{\mu}$ then $(\zeta, v) = \big(0, t(1 - \overline{\mu}^2)^{-1}\big)$ is a fixed point of T (note that, when $\mu t = t\overline{\mu}$, if t is pure imaginary then so is $t(1 - \overline{\mu}^2)^{-1}$).

The action of T on $\overline{\mathbf{H}^n_{\mathbb{H}}} - \{\infty\}$ is given by

$$T(\zeta, v, u) = \Big(U\zeta\overline{\mu} + \tau,\ t + \mu v\overline{\mu} - 2\mathrm{Im}(\tau^*\mu\zeta\overline{\mu}),\ u\Big).$$

Observe that T maps the horosphere of height $u \in [0, \infty)$ to itself. The Cygan translation length of T at (ζ, v), denoted $\ell_T(\zeta, v) = d_H\big(T(\zeta, v), (\zeta, v)\big) = d_H\big(T(\zeta, v, u), (\zeta, v, u)\big)$, is:

$$\begin{aligned} \ell_T(\zeta, v) &= \left|\Big(U\zeta\overline{\mu} + \tau - \zeta,\ t + \mu v\overline{\mu} - v + 2\mathrm{Im}\big((\zeta^* - \tau^*)(U\zeta\overline{\mu} + \tau)\big)\Big)\right|_H \\ &= \Big|2\zeta^* U\zeta\overline{\mu} - 2\tau^*\mu\zeta\overline{\mu} + 2\zeta^*\tau - \|\tau\|^2 + t - 2\|\zeta\|^2 + \mu v\overline{\mu} - v\Big|^{1/2}. \end{aligned} \tag{16}$$

The vertical projection of T acting on $\mathbb{H}^{n-1}$ is $\zeta \longmapsto U\zeta\overline{\mu} + \tau$. Its Euclidean translation length is $\|\Pi T(\zeta, v) - \Pi(\zeta, v)\| = \|U\zeta\overline{\mu} + \tau - \zeta\|$. It is easy to show:

Corollary 2.4 *Let $(\zeta, v) \in \mathfrak{N}_{4n-1}$ and let $\Pi : \mathfrak{N}_{4n-1} \longrightarrow \mathbb{H}^{n-1}$ be vertical projection given by $\Pi : (\zeta, v) \longmapsto \zeta$. If T is given by (1) then*

$$\big\|\Pi T(\zeta, v) - \Pi(\zeta, v)\big\| \leq \ell_T(\zeta, v).$$

The following proposition relates the Cygan translation lengths of T at two points of $\mathfrak{N}_{4n-1}$. It is a generalisation of Lemma 1.5 of [21].

Proposition 2.5 *Let T be given by* (1). *Let (ζ, v) and (ξ, r) be two points in $\mathfrak{N}_{4n-1}$. Write $(\zeta, v)^{-1}(\xi, r) = (\eta, s)$. Then*

$$\ell_T(\xi, r)^2 \leq \ell_T(\zeta, v)^2 + 4\|\Pi T(\zeta, v) - \Pi(\zeta, v)\| \, \|\eta\| + 2N_{U,\mu}\|\eta\|^2 + N_\mu|s|.$$

Proof: We write $(\xi, r) = (\zeta, v)(\eta, s) = (\zeta + \eta,\, v + s + \eta^*\zeta - \zeta^*\eta)$. Then

$$\begin{aligned}
&2\xi^*U\xi\overline{\mu} - 2\tau^*\mu\xi\overline{\mu} + 2\xi^*\tau - \|\tau\|^2 + t - 2\|\xi\|^2 + \mu r\overline{\mu} - r \\
&\quad = \; 2(\zeta+\eta)^*U(\zeta+\eta)\overline{\mu} - 2\tau^*\mu(\zeta+\eta)\overline{\mu} + 2(\zeta+\eta)^*\tau - \|\tau\|^2 + t \\
&\qquad\quad -2\|\zeta+\eta\|^2 + \mu(v + s + \eta^*\zeta - \zeta^*\eta)\overline{\mu} - v - s - \eta^*\zeta + \zeta^*\eta \\
&\quad = \; 2\zeta^*U\zeta\overline{\mu} - 2\tau^*\mu\zeta\overline{\mu} + 2\zeta^*\tau - \|\tau\|^2 + t - 2\|\zeta\|^2 + \mu v\overline{\mu} - v \\
&\qquad\quad +2\eta^*(U\zeta\overline{\mu} + \tau - \zeta) - 2(\mu\zeta^*U^* + \tau^* - \zeta^*)U\eta\overline{\mu} + 2\eta^*(U\eta - \eta\mu)\overline{\mu} + (\mu s - s\mu)\overline{\mu}.
\end{aligned}$$

Therefore, using (16),

$$\begin{aligned}
\ell_T(\xi, r)^2 &= \Big|2\xi^*U\xi\overline{\mu} - 2\tau^*\mu\xi\overline{\mu} + 2\xi^*\tau - \|\tau\|^2 + t - 2\|\xi\|^2 + \mu r\overline{\mu} - r\Big| \\
&\leq \Big|2\zeta^*U\zeta\overline{\mu} - 2\tau^*\mu\zeta\overline{\mu} + 2\zeta^*\tau - \|\tau\|^2 + t - 2\|\zeta\|^2 + \mu v\overline{\mu} - v\Big| \\
&\qquad +2\Big|\eta^*(U\zeta\overline{\mu} + \tau - \zeta)\Big| + 2\Big|(\mu\zeta^*U^* + \tau^* - \zeta^*)U\eta\overline{\mu}\Big| \\
&\qquad +2\|\eta\| \, \|U\eta\overline{\mu} - \eta\| + |\mu s - s\mu| \\
&\leq \ell_T(\zeta, v)^2 + 4\|\eta\| \, \|U\zeta\overline{\mu} + \tau - \zeta\| + 2N_{U,\mu}\|\eta\|^2 + N_\mu|s|.
\end{aligned}$$

The result follows since $U\zeta\overline{\mu} + \tau - \zeta = \Pi T(\zeta, v) - \Pi(\zeta, v)$. □

3 A precisely invariant sub-horospherical region

In this section we show how Theorem 1.3 follows from Theorem 1.1. This argument follows Lemmas 3.3 and 3.4 of [21].

Proof: (Theorem 1.3.) Let $z = (\zeta, v, u)$ be any point on the Cygan sphere with radius r and centre $(\zeta_0, v_0, 0) = (\zeta_0, v_0) \in \mathfrak{N}_{4n-1} \subset \partial\mathbf{H}^n_{\mathbb{H}}$ and write $(\eta, s) = (\zeta, v)^{-1}(\zeta_0, v_0)$. Then we have

$$r^2 = d_H\big((\zeta, v, u), (\zeta_0, v_0, 0)\big)^2 = \Big|\|\eta\|^2 + u + s\Big| = \Big(\big(\|\eta\| + u\big)^2 + |s|^2\Big)^{1/2}.$$

In particular, $r^2 \geq \|\eta\|^2 + u$ and $r^2 \geq |s|$. We claim that the Cygan sphere with centre (ζ_0, v_0) and radius r does not intersect $\mathcal{U}_T$ when r satisfies:

$$r^2 \leq \frac{\ell_T(\zeta_0, v_0)^2}{K} + \frac{4\|\Pi T(\zeta_0, v_0) - \Pi(\zeta_0, v_0)\|^2}{K(K - 2N_{U,\mu})}. \tag{17}$$

To see this, using Proposition 2.5 to compare $\ell_T(\zeta_0, v_0)$ with $\ell_T(\zeta, v) = \ell_T(z)$, we have

$$\begin{aligned}
u &\leq r^2 - \|\eta\|^2 \\
&= \frac{K}{K-N_\mu}\, r^2 - \frac{N_\mu}{K-N_\mu}\, r^2 - \|\eta\|^2 \\
&\leq \frac{K}{K-N_\mu}\left(\frac{\ell_T(\zeta_0,v_0)^2}{K} + \frac{4\|\Pi T(\zeta_0,v_0) - \Pi(\zeta_0,v_0)\|^2}{K(K-2N_{U,\mu})}\right) - \frac{N_\mu}{K-N_\mu}\,|s| - \|\eta\|^2 \\
&\leq \frac{1}{K-N_\mu}\Big(\ell_T(z)^2 + 4\|\Pi T(z) - \Pi(z)\|\,\|\eta\| + 2N_{U,\mu}\|\eta\|^2 + N_\mu|s|\Big) \\
&\quad + \frac{4}{(K-N_\mu)(K-2N_{U,\mu})}\Big(\|\Pi T(z) - \Pi(z)\| + N_{U,\mu}\|\eta\|\Big)^2 - \frac{N_\mu}{K-N_\mu}\,|s| - \|\eta\|^2 \\
&= \frac{\ell_T(z)^2}{K-N_\mu} + \frac{4\|\Pi T(z)-\Pi(z)\|^2}{(K-N_\mu)(K-2N_{U,\mu})} + \frac{4K\|\Pi T(z) - \Pi(z)\|}{(K-N_\mu)(K-2N_{U,\mu})}\,\|\eta\| \\
&\quad - \frac{(K-N_\mu)(K-2N_{U,\mu}) - 2N_{U,\mu}K}{(K-N_\mu)(K-2N_{U,\mu})}\,\|\eta\|^2 \\
&\leq \frac{\ell_T(z)^2}{K-N_\mu} + \frac{4(2K-N_\mu)\|\Pi T(z)-\Pi(z)\|^2}{(K-N_\mu)\big((K-N_\mu)(K-2N_{U,\mu}) - 2N_{U,\mu}K\big)},
\end{aligned}$$

where the last inequality follows by finding the value of $\|\eta\|$ maximising the previous line. Hence, when r satisfies (17) the Cygan sphere with centre (ζ_0, v_0) and radius r lies outside $\mathcal{U}_T$.

Now suppose that the radius r_S of isometric sphere of S satisfies the bound (6). Consider the Cygan sphere with centre $S^{-1}(\infty) = (\zeta_0, v_0)$ and radius r with equality in (17). That is

$$r^2 = \frac{\ell_T(\zeta_0,v_0)^2}{K} + \frac{4\|\Pi T(\zeta_0,v_0) - \Pi(\zeta_0,v_0)\|^2}{K(K-2N_{U,\mu})}. \tag{18}$$

We know that S sends this sphere to the Cygan sphere with centre $S(\infty) = (\widetilde{\zeta}_0, \widetilde{v}_0)$ and radius $\widetilde{r} = r_S^2/r$. We claim that $\widetilde{r}$ satisfies (17). It will follow from this claim that both spheres are disjoint from $\mathcal{U}_T$. Since S sends the exterior of the first sphere to the interior of the second, it will follow that $S(\mathcal{U}_T) \cap \mathcal{U}_T = \emptyset$.

In order to verify the claim, use (18) and (6) to check that:

$$\begin{aligned}
\widetilde{r}^2 &= r_S^4/r^2 \\
&\leq \frac{1}{r^2}\left(\frac{\ell_T(\zeta_0,v_0)\ell_T(\widetilde{\zeta}_0,\widetilde{v}_0)}{K} + \frac{4\big\|\Pi T(\zeta_0,v_0) - \Pi(\zeta_0,v_0)\big\|\,\big\|\Pi T(\widetilde{\zeta}_0,\widetilde{v}_0) - \Pi(\widetilde{\zeta}_0,\widetilde{v}_0)\big\|}{K(K-2N_{U,\mu})}\right)^2 \\
&\leq \left(\frac{\ell_T(\widetilde{\zeta}_0,\widetilde{v}_0)^2}{K} + \frac{4\|\Pi T(\widetilde{\zeta}_0,\widetilde{v}_0) - \Pi(\widetilde{\zeta}_0,\widetilde{v}_0)\|^2}{K(K-2N_{U,\mu})}\right).
\end{aligned}$$

Thus $\widetilde{r}$ satisfies (17) as claimed.

Therefore, if $S \in G - G_\infty$ then the image of $\mathcal{U}_T$ does not intersect its image under S. On the other hand, clearly T maps $\mathcal{U}_T$ to itself. Thus every element of $G_\infty = \langle T \rangle$ maps $\mathcal{U}_T$ to itself. Hence $\mathcal{U}_T$ is precisely invariant under G_∞ in G. This proves Theorem 1.3. $\square$

4 The dynamical system involving S and T

4.1 The series $S_{j+1} = S_j T S_j^{-1}$

We begin by developing some recurrence relations with $S_0 = S$ and $S_{j+1} = S_j T S_j^{-1}$ with T in the normal form (1). Write S_j in the standard form (9) with subscript j. Then S_{j+1} is

$$\begin{pmatrix} a_{j+1} & \gamma_{j+1}^* & b_{j+1} \\ \alpha_{j+1} & A_{j+1} & \beta_{j+1} \\ c_{j+1} & \delta_{j+1}^* & d_{j+1} \end{pmatrix} = \begin{pmatrix} a_j & \gamma_j^* & b_j \\ \alpha_j & A_j & \beta_j \\ c_j & \delta_j^* & d_j \end{pmatrix} \begin{pmatrix} \mu & -\sqrt{2}\tau^*\mu & (-\|\tau\|^2+t)\mu \\ 0 & U & \sqrt{2}\tau\mu \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \overline{d}_j & \beta_j^* & \overline{b}_j \\ \delta_j & A_j^* & \gamma_j \\ \overline{c}_j & \alpha_j^* & \overline{a}_j \end{pmatrix}. \tag{19}$$

Performing the matrix multiplication of (19), we have the recurrence relations:

$$\begin{aligned}
a_{j+1} &= \gamma_j^* U \delta_j - \sqrt{2} a_j \tau^* \mu \delta_j + \sqrt{2}\gamma_j^* \tau \mu \overline{c}_j - a_j\big(\|\tau\|^2 - t\big)\mu \overline{c}_j + a_j \mu \overline{d}_j + b_j \mu \overline{c}_j, & (20)\\
\gamma_{j+1} &= A_j U^* \gamma_j - \sqrt{2} A_j \overline{\mu} \tau \overline{a}_j + \sqrt{2}\alpha_j \overline{\mu} \tau^* \gamma_j - \alpha_j \overline{\mu}(\|\tau\|^2 + t)\overline{a}_j + \alpha_j \overline{\mu}\, \overline{b}_j + \beta_j \overline{\mu}\, \overline{a}_j, & (21)\\
b_{j+1} &= \gamma_j^* U \gamma_j - \sqrt{2} a_j \tau^* \mu \gamma_j + \sqrt{2}\gamma_j^* \tau \mu \overline{a}_j - a_j(\|\tau\|^2 - t)\mu \overline{a}_j + a_j \mu \overline{b}_j + b_j \mu \overline{a}_j, & (22)\\
\alpha_{j+1} &= A_j U \delta_j - \sqrt{2}\alpha_j \tau^* \mu \delta_j + \sqrt{2} A_j \tau \mu \overline{c}_j - \alpha_j(\|\tau\|^2 - t)\mu \overline{c}_j + \alpha_j \mu \overline{d}_j + \beta_j \mu \overline{c}_j, & (23)\\
A_{j+1} &= A_j U A_j^* - \sqrt{2}\alpha_j \tau^* \mu A_j^* + \sqrt{2} A_j \tau \mu \alpha_j^* - \alpha_j(\|\tau\|^2 - t)\mu \alpha_j^* + \alpha_j \mu \beta_j^* + \beta_j \mu \alpha_j^*, & (24)\\
\beta_{j+1} &= A_j U \gamma_j - \sqrt{2}\alpha_j \tau^* \mu \gamma_j + \sqrt{2} A_j \tau \mu \overline{a}_j - \alpha_j(\|\tau\|^2 - t)\mu \overline{a}_j + \alpha_j \mu \overline{b}_j + \beta_j \mu \overline{a}_j, & (25)\\
c_{j+1} &= \delta_j^* U \delta_j - \sqrt{2} c_j \tau^* \mu \delta_j + \sqrt{2}\delta_j^* \tau \mu \overline{c}_j - c_j(\|\tau\|^2 - t)\mu \overline{c}_j + c_j \mu \overline{d}_j + d_j \mu \overline{c}_j, & (26)\\
\delta_{j+1} &= A_j U^* \delta_j - \sqrt{2} A_j \overline{\mu} \tau \overline{c}_j + \sqrt{2}\alpha_j \overline{\mu} \tau^* \delta_j - \alpha_j \overline{\mu}(\|\tau\|^2 + t)\overline{c}_j + \beta_j \overline{\mu}\, \overline{c}_j + \alpha_j \overline{\mu}\, \overline{d}_j, & (27)\\
d_{j+1} &= \delta_j^* U \gamma_j - \sqrt{2} c_j \tau^* \mu \gamma_j + \sqrt{2}\delta_j^* \tau \mu \overline{a}_j - c_j\big(\|\tau\|^2 - t\big)\mu \overline{a}_j + c_j \mu \overline{b}_j + d_j \mu \overline{a}_j. & (28)
\end{aligned}$$

We also define $\widetilde{S}_{j+1} = S_j^{-1} T S_j$ and we denote its entries $\widetilde{a}_{j+1}$ and so on. We will only need

$$\widetilde{c}_{j+1} = \alpha_j^* U \alpha_j - \sqrt{2}\overline{c}_j \tau^* \mu \alpha_j + \sqrt{2}\alpha_j^* \tau \mu c_j - \overline{c}_j(\|\tau\|^2 - t)\mu c_j + \overline{c}_j \mu a_j + \overline{a}_j \mu c_j. \tag{29}$$

Suppose $S_j^{-1}(\infty)$ and $S_j(\infty)$ have Heisenberg coordinates (ζ_j, r_j) and (ω_j, s_j) respectively. So:

$$S_j^{-1}(\infty) = \begin{pmatrix} -\|\zeta_j\|^2 + r_j \\ \sqrt{2}\zeta_j \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{d}_j \overline{c}_j^{-1} \\ \delta_j \overline{c}_j^{-1} \\ 1 \end{pmatrix}, \qquad S_j(\infty) = \begin{pmatrix} -\|\omega_j\|^2 + s_j \\ \sqrt{2}\omega_j \\ 1 \end{pmatrix} = \begin{pmatrix} a_j c_j^{-1} \\ \alpha_j c_j^{-1} \\ 1 \end{pmatrix}. \tag{30}$$

Then $\Pi(S^{-1}(\infty)) = \zeta_j$, $\Pi(TS^{-1}(\infty)) = U\zeta_j\overline{\mu} + \tau$, $\Pi(S\infty)) = \omega_j$ and $\Pi(TS(\infty)) = U\omega_j\overline{\mu} + \tau$. We also define

$$\begin{aligned}
\xi_j &:= \Pi(TS^{-1}(\infty)) - \Pi(S^{-1}(\infty)) = U\zeta_j\overline{\mu} + \tau - \zeta_j = \frac{1}{\sqrt{2}}\Big(U\delta_j \overline{c}_j^{-1}\overline{\mu} - \delta_j \overline{c}_j^{-1}\Big) + \tau, & (31)\\
\eta_j &:= \Pi(TS(\infty)) - \Pi(S(\infty)) = U\omega_j\overline{\mu} + \tau - \omega_j = \frac{1}{\sqrt{2}}\Big(U\alpha_j c_j^{-1}\overline{\mu} - \alpha_j c_j^{-1}\Big) + \tau, & (32)\\
B_j &:= A_j - \alpha_j c_j^{-1}\delta_j^*. & (33)
\end{aligned}$$

Note that Lemma 2.1 implies $B_j \in \mathrm{Sp}(n-1)$. Also, using (26) and (29) we have

$$\begin{aligned}
c_j^{-1} c_{j+1} \overline{c}_j^{-1} &= 2\zeta_j^* U \zeta_j - 2\tau^* \mu \zeta_j + 2\zeta_j^* \tau \mu - \|\tau\|^2\mu + t\mu - 2\|\zeta_j\|^2\mu + \mu r_j - r_j \mu,\\
\overline{c}_j^{-1} \widetilde{c}_{j+1} c_j^{-1} &= 2\omega_j^* U \omega_j - 2\tau^* \mu \omega_j + 2\omega_j^* \tau \mu - \|\tau\|^2\mu + t\mu - 2\|\omega_j\|^2\mu + \mu s_j - s_j \mu.
\end{aligned}$$

Lemma 4.1 *If c_j, $\widetilde{c}_j$, ξ_j and η_j are given by (26), (29), (31) and (32), then*

$$0 = 2\|\xi_j\|^2 + 2\mathrm{Re}(c_j^{-1}c_{j+1}\overline{c}_j^{-1}\overline{\mu}), \quad 0 = 2\|\eta_j\|^2 + 2\mathrm{Re}(\overline{c}_j^{-1}\widetilde{c}_{j+1}c_j^{-1}\overline{\mu}).$$

Proof: We only prove the first identity. Writing out $2\mathrm{Re}(c_j^{-1}c_{j+1}\overline{c}_j^{-1}\overline{\mu})$ we obtain

$$\begin{aligned} 2\mathrm{Re}(c_j^{-1}c_{j+1}\overline{c}_j^{-1}\overline{\mu}) &= 2\zeta_j^*U\zeta_j\overline{\mu} - 2\tau^*\mu\zeta_j\overline{\mu} + 2\zeta_j^*\tau - \|\tau\|^2 + t - 2\|\zeta_j\|^2 + \mu r_j\overline{\mu} - r_j \\ &\quad +2\mu\zeta_j^*U^*\zeta_j - 2\mu\zeta_j^*\overline{\mu}\tau + 2\tau^*\zeta_j - \|\tau\|^2 - t - 2\|\zeta_j\|^2 - \mu r_j\overline{\mu} + r_j \\ &= -2\big(\mu\zeta_j^*U^* + \tau^* - \zeta_j^*\big)(U\zeta_j\overline{\mu} + \tau - \zeta_j), \end{aligned}$$

where we have used $\tau^*\mu = \tau^*U$. The result follows since $\xi_j = U\zeta_j\overline{\mu} + \tau - \zeta_j$. □

Lemma 4.2 *Let $S_j^{-1}(\infty) = (\zeta_j, r_j)$ and $S_j(\infty) = (\omega_j, s_j)$. Let ξ_j and η_j be given by (31) and (32). Then*

$$\zeta_{j+1} = \frac{1}{\sqrt{2}}\delta_{j+1}\overline{c}_{j+1}^{-1} = \omega_j - B_jU^*\xi_j\overline{c}_j\overline{c}_{j+1}^{-1}, \tag{34}$$

$$-\|\zeta_{j+1}\|^2 + r_{j+1} = \overline{d}_{j+1}\overline{c}_{j+1}^{-1} = -\|\omega_j\|^2 + s_j + \overline{c}_j^{-1}\overline{\mu}\,\overline{c}_j\overline{c}_{j+1}^{-1} + 2\omega_j^*(B_jU^*\xi_j\overline{c}_j\overline{c}_{j+1}^{-1}), \tag{35}$$

$$\omega_{j+1} = \frac{1}{\sqrt{2}}\alpha_{j+1}c_{j+1}^{-1} = \omega_j + B_j\xi_j\mu\overline{c}_jc_{j+1}^{-1}, \tag{36}$$

$$-\|\omega_{j+1}\|^2 + s_{j+1} = a_{j+1}c_{j+1}^{-1} = -\|\omega_j\|^2 + s_j + \overline{c}_j^{-1}\mu\overline{c}_jc_{j+1}^{-1} - 2\omega_j^*(B_j\xi_j\mu\overline{c}_jc_{j+1}^{-1}). \tag{37}$$

In particular,

$$\xi_{j+1} = U\zeta_{j+1}\overline{\mu} + \tau - \zeta_{j+1} = \eta_j - U(B_jU^*\xi_j\overline{c}_j\overline{c}_{j+1}^{-1})\overline{\mu} + (B_jU^*\xi_j\overline{c}_j\overline{c}_{j+1}^{-1}), \tag{38}$$

$$\eta_{j+1} = \eta_j + U\omega_{j+1}\overline{\mu} + \tau - \omega_{j+1} = U(B_j\xi_j\mu\overline{c}_jc_{j+1}^{-1})\overline{\mu} - (B_j\xi_j\mu\overline{c}_jc_{j+1}^{-1}). \tag{39}$$

Proof: We have

$$\begin{aligned} a_{j+1} &= \gamma_j^*U\delta_j - \sqrt{2}a_j\tau^*\mu\delta_j + \sqrt{2}\gamma_j^*\tau\mu\overline{c}_j - a_j\big(\|\tau\|^2 - t\big)\mu\overline{c}_j + a_j\mu\overline{d}_j + b_j\mu\overline{c}_j \\ &= a_jc_j^{-1}c_{j+1} + (\gamma_j^* - a_jc_j^{-1}\delta_j^*)(U\delta_j\overline{c}_j^{-1}\overline{\mu} - \delta_j\overline{c}_j^{-1} + \sqrt{2}\tau)\mu\overline{c}_j \\ &\quad +(\gamma_j^*\delta_j\overline{c}_j^{-1} - a_jc_j^{-1}\delta_j^*\delta_j\overline{c}_j^{-1} + b_j - a_jc_j^{-1}d_j)\mu\overline{c}_j, \\ &= a_jc_j^{-1}c_{j+1} + \overline{c}_j^{-1}\mu\overline{c}_j - \overline{c}_j^{-1}\alpha_j^*B_j(U\delta_j\overline{c}_j^{-1}\overline{\mu} - \delta_j\overline{c}_j^{-1} + \sqrt{2}\tau)\mu\overline{c}_j. \end{aligned}$$

On the last line we used (10) and (15) to substitute for $\gamma_j^*\delta_j$ and $\delta_j^*\delta_j$ and Lemma 2.1 to write $\gamma_j^* - a_jc_j^{-1}\delta_j^* = -\overline{c}_j^{-1}\alpha_j^*B_j$. Now using the definitions of s_j, ω_j and ξ_j from (30) and (31) we obtain (39).

The other identities follow similarly. When proving the identities for ζ_{j+1} and $-\|\zeta_{j+1}\|^2 + r_{j+1}$ we also use $U^*\tau = \overline{\mu}\tau$. □

The following corollary, along with Proposition 2.5, will enable us to compare the Cygan translation length of T at $S_{j+1}^{-1}(\infty)$ and $S_{j+1}(\infty)$ with its Cygan translation lengths at $S_j^{-1}(\infty)$ and $S_j(\infty)$.

Corollary 4.3 *Write $S_j^{-1}(\infty) = (\zeta_j, r_j)$ and $S_j(\infty) = (\omega_j, s_j)$ in Heisenberg coordinates. Then*

$$\begin{aligned} (\omega_j, s_j)^{-1}(\zeta_{j+1}, r_{j+1}) &= \Big(-B_jU^*\xi_j\overline{c}_j\overline{c}_{j+1}^{-1},\ \mathrm{Im}\big(\overline{c}_j^{-1}\overline{\mu}\,\overline{c}_j\overline{c}_{j+1}^{-1}\big)\Big), \\ (\omega_j, s_j)^{-1}(\omega_{j+1}, s_{j+1}) &= \Big(B_j\xi_j\mu\overline{c}_jc_{j+1}^{-1},\ \mathrm{Im}\big(\overline{c}_j^{-1}\mu\overline{c}_jc_{j+1}^{-1}\big)\Big). \end{aligned}$$

4.2 Translation lengths of T at $S_j^{-1}(\infty)$ and $S_j(\infty)$

We are now ready to define the main quantities which we use for defining the recurrence relation between S_{j+1} and S_j. They are defined by:

$$X_j = \frac{\ell_T(S_j^{-1}(\infty))}{r_{S_j}}, \qquad Y_j = \frac{\big\|\Pi T S_j^{-1}(\infty) - \Pi S_j^{-1}(\infty)\big\|}{r_{S_j}},$$
$$\widetilde{X}_j = \frac{\ell_T(S_j(\infty))}{r_{S_j}}, \qquad \widetilde{Y}_j = \frac{\big\|\Pi T S_j(\infty) - \Pi S_j(\infty)\big\|}{r_{S_j}}.$$

Observe that Corollary 2.4 immediately implies $Y_j \le X_j$ and $\widetilde{Y}_j \le \widetilde{X}_j$. Using (16), we see that in terms of the matrix entries they are given by:

$$\begin{aligned} X_j^2 &= |c_j^{-1}c_{j+1}\overline{c}_j^{-1}|\,|c_j| \\ &= \Big|2\zeta_j^*U\zeta_j - 2\tau^*\mu\zeta_j + 2\zeta_j^*\tau\mu - (\|\tau\|^2 - t)\mu - 2\|\zeta_j\|^2\mu + \mu r_j - r_j\mu\Big|\,|c_j|, \qquad (40)\\ \widetilde{X}_j^2 &= |\overline{c}_j^{-1}\widetilde{c}_{j+1}c_j^{-1}|\,|c_j| \\ &= \Big|2\omega_j^*U\omega_j - 2\tau^*\mu\omega_j + 2\omega_j^*\tau\mu - (\|\tau\|^2 - t)\mu - 2\|\omega_j\|^2\mu + \mu s_j - s_j\mu\Big|\,|c_j|, \qquad (41)\\ Y_j^2 &= \|\xi_j\|^2|c_j| \;=\; \|U\zeta_j\overline{\mu} + \tau - \zeta_j\|^2|c_j|, \qquad (42)\\ \widetilde{Y}_j^2 &= \|\eta_j\|^2|c_j| \;=\; \|U\omega_j\overline{\mu} + \tau - \omega_j\|^2|c_j|. \qquad (43) \end{aligned}$$

In Section 6 we will show (6) implies that the sequence $S_{j+1} = S_jTS_j^{-1}$ converges to T in the topology induced by the Hilbert-Schmidt norm on $\mathrm{PSp}(n,1)$. To do so, we need the following two lemmas giving X_{j+1}, $\widetilde{X}_{j+1}$, Y_{j+1} and $\widetilde{Y}_{j+1}$ in terms of X_j, $\widetilde{X}_j$, Y_j and $\widetilde{Y}_j$.

Lemma 4.4 *We claim that*

$$\begin{aligned} X_{j+1}^2 &\le X_j^2\widetilde{X}_j^2 + 4Y_j\widetilde{Y}_j + 2N_{U,\mu} + N_\mu, \qquad (44)\\ \widetilde{X}_{j+1}^2 &\le X_j^2\widetilde{X}_j^2 + 4Y_j\widetilde{Y}_j + 2N_{U,\mu} + N_\mu. \qquad (45) \end{aligned}$$

Proof: Writing $S_j^{-1}(\infty)$ and $S_j(\infty)$ in Heisenberg coordinates and using Proposition 2.5 and Corollary 4.3, we have

$$\begin{aligned} &\ell_T\big(S_{j+1}^{-1}(\infty)\big)^2 \\ &\le \ell_T\big(S_j(\infty)\big)^2 + 4\big\|\Pi TS_j(\infty) - S_j(\infty)\big\|\,\big\|-B_jU^*\xi_j\overline{c}_j\overline{c}_{j+1}^{-1}\big\| \\ &\qquad +2N_{U,\mu}\big\|-B_jU^*\xi_j\overline{c}_j\overline{c}_{j+1}^{-1}\|^2 + N_\mu\big|\mathrm{Im}\big(\overline{c}_j^{-1}\overline{\mu}\,c_j\overline{c}_{j+1}^{-1}\big)\big| \\ &\le \ell_T\big(S_j(\infty)\big)^2 + 4\|\eta_j\|\,\|\xi_j\|\,|c_j|\,|c_{j+1}|^{-1} + 2N_{U,\mu}\|\xi_j\|^2|c_j|^2|c_{j+1}|^{-2} + N_\mu|c_{j+1}|^{-1}. \end{aligned}$$

Now, multiply on the left and right by $|c_{j+1}| = 1/r_{S_{j+1}}^2$ and use $\ell_T(S_j^{-1}(\infty)) = X_jr_{S_j}$ and $\ell_T(S_j(\infty)) = \widetilde{X}_jr_{S_j}$. This gives

$$X_{j+1}^2 \le \widetilde{X}_j^2|c_{j+1}|\,|c_j|^{-1} + 4\|\eta_j\|\,\|\xi_j\|\,|c_j| + 2N_{U,\mu}\|\xi_j\|^2|c_j|^2|c_{j+1}|^{-1} + N_\mu.$$

Finally, we use $|c_{j+1}|\,|c_j|^{-1} = X_j^2$, $\|\xi_j\|\,|c_j|^{1/2} = Y_j$ and $\|\eta_j\|\,|c_j|^{1/2} = \widetilde{Y}_j$. This gives

$$X_{j+1}^2 \le X_j^2\widetilde{X}_j^2 + 4Y_j\widetilde{Y}_j + 2N_{U,\mu}Y_j^2X_j^{-2} + N_\mu.$$

The inequality (44) follows since $Y_j \le X_j$. The inequality (45) follows similarly. $\square$

We now estimate Y_{j+1} and $\widetilde{Y}_{j+1}$ in terms of X_j, $\widetilde{X}_j$ and so on.

Lemma 4.5 *We claim that*

$$\begin{aligned} Y_{j+1}^2 &\le \widetilde{Y}_j^2 X_j^2 + 2N_{U,\mu} Y_j \widetilde{Y}_j + N_{U,\mu}^2, &(46)\\ \widetilde{Y}_{j+1}^2 &\le \widetilde{Y}_j^2 X_j^2 + 2N_{U,\mu} Y_j \widetilde{Y}_j + N_{U,\mu}^2. &(47)\end{aligned}$$

Proof: Using the definition of Y_j from (42) and the identity for ξ_{j+1} from (38), we have:

$$\begin{aligned} Y_{j+1} &= \|\xi_{j+1}\|\,|c_{j+1}|^{1/2}\\ &= \left\|\eta_j - U(B_j U^*\xi_j \bar{c}_j \bar{c}_{j+1}^{-1})\overline{\mu} + (B_j U^* \xi_j \bar{c}_j \bar{c}_{j+1}^{-1})\right\|\,|c_{j+1}|^{1/2}\\ &\le \widetilde{Y}_j |c_j|^{-1/2}|c_{j+1}|^{1/2} + N_{U,\mu} Y_j |c_j|^{1/2}|c_{j+1}|^{-1/2}\\ &= \widetilde{Y}_j X_j + N_{U,\mu} Y_j X_j^{-1}. \end{aligned}$$

Squaring and using $Y_j \le X_j$ gives (46). A similar argument gives the inequality (47). $\square$

Therefore we have recurrence relations bounding X_{j+1}, $\widetilde{X}_{j+1}$, Y_{j+1} and $\widetilde{Y}_{j+1}$ (that is translation lengths and radii) in terms of the same quantities for the index j. In the next section, we find a basin of attraction for this dynamical system.

5 Convergence of the dynamical system

In this section we interpret the condition (6) of Theorem 1.1 in terms of our dynamical system involving translation lengths, and we show that (6) means that X_j, $\widetilde{X}_j$, Y_j and $\widetilde{Y}_j$ converge. Broadly speaking the argument will be based on the argument of Parker in [21] for subgroups of $\mathrm{SU}(n,1)$ containing a Heisenberg translation. This argument was used by Kim and Parker in [16] for subgroups of $\mathrm{Sp}(n,1)$ containing a Heisenberg translation. For Heisenberg translations $N_{U,\mu} = N_\mu = 0$, meaning that the inequalities from Lemmas 4.4 and 4.5 are much simpler (see page 307 of [16]). Our proofs in the case when $U \ne I_{n-1}$ or $\mu \ne 1$ are more complicated.

Recall the definition of K from (5). The only properties of K that we need are that $(1+2N_{U,\mu})/2 < K < 1 - 2N_{U,\mu} < 1$ and that K satisfies the equation:

$$(K - 2N_{U,\mu})(1-K) = 2N_{U,\mu} + N_\mu. \tag{48}$$

Observe that (44), (45), (46) and (47) together with (48) imply

$$\begin{aligned} \max\{X_{j+1}^2, \widetilde{X}_{j+1}^2\} &\le X_j^2 \widetilde{X}_j^2 + 4Y_j\widetilde{Y}_j + (K-2N_{U,\mu})(1-K), &(49)\\ \max\{Y_{j+1}^2, \widetilde{Y}_{j+1}^2\} &\le X_j^2 \widetilde{Y}_j^2 + 2N_{U,\mu}Y_j\widetilde{Y}_j + N_{U,\mu}(K-2N_{U,\mu})(1-K)/2. &(50)\end{aligned}$$

Our goal in this section is to prove the following theorem.

Theorem 5.1 *Suppose X_j, $\widetilde{X}_j$, Y_j and $\widetilde{Y}_j$ satisfy* (49) *and* (50). *If*

$$X_0\widetilde{X}_0 + \frac{4Y_0\widetilde{Y}_0}{K - 2N_{U,\mu}} < K. \tag{51}$$

then for all $\varepsilon > 0$ there exists J_ε so that for all $j \ge J_\varepsilon$:

$$\max\{X_j^2, \widetilde{X}_j^2\} < 1 - K + \varepsilon, \quad \max\{Y_j^2, \widetilde{Y}_j^2\} < N_{U,\mu}(1-K)/2 + \varepsilon. \tag{52}$$

Note that (51) is simply the statement that (6) fails written in terms of X_0, $\widetilde{X}_0$, Y_0 and $\widetilde{Y}_0$. Moreover, the case of Theorem 5.1 when $N_{U,\mu} = 0$ is a restatement of Lemma 2.6 of [21] (see also [16]) and is proved similarly. Therefore we only consider the case $N_{U,\mu} > 0$ is what follows. Our strategy for proving Theorem 5.1 will have three steps:

- We first show that (51) implies that for all $j \geq 1$

$$\max\{X_j^2, \widetilde{X}_j^2\} + \frac{4\max\{Y_j^2, \widetilde{Y}_j^2\}}{K - 2N_{U,\mu}} < K.$$

- Then we show that for sufficiently large j we have

$$\max\{X_j^2, \widetilde{X}_j^2\} < K - 2N_{U,\mu}, \quad \max\{Y_j^2, \widetilde{Y}_j^2\} < N_{U,\mu}(K - 2N_{U,\mu})/2. \tag{53}$$

- Finally we show that for all $\varepsilon > 0$ there exists J_ε so that (52) holds for all $j \geq J_\varepsilon$.

We will need to work quite hard to obtain these bounds. In part, this is because the bound for $\max\{Y_j^2, \widetilde{Y}_j^2\}$ in (53) is clearly false in the case $N_{U,\mu} = 0$. The place we use $N_{U,\mu} > 0$ is in the proof of Proposition 5.6.

5.1 The initial bound

For $j \geq 1$ define

$$Z_j = \max\{X_j^2, \widetilde{X}_j^2\} + \frac{4\max\{Y_j^2, \widetilde{Y}_j^2\}}{K - 2N_{U,\mu}}. \tag{54}$$

Observe that (49) and (50) imply

$$Z_{j+1} \leq X_j^2 Z_j + \frac{4KY_j\widetilde{Y}_j}{K - 2N_{U,\mu}} + K(1-K). \tag{55}$$

We show that $Z_j < K$ for all $j \geq 1$.

Lemma 5.2 *Suppose that* (51) *holds. Then* $Z_j - K \leq K^{j-1}(Z_1 - K) < 0$ *for all* $j \geq 1$.

Proof: Suppose that (51) holds. Interchanging S_0 and S_0^{-1} if necessary, we also suppose that $X_0\widetilde{Y}_0 \leq \widetilde{X}_0 Y_0$. We first show that $Z_1 < K$.

$$\begin{aligned}
Z_1 &\leq \left(X_0^2\widetilde{X}_0^2 + 4Y_0\widetilde{Y}_0 + 2N_{U,\mu} + N_\mu\right) + \left(X_0^2\widetilde{Y}_0^2 + 2N_{U,\mu}Y_0\widetilde{Y}_0 + N_{U,\mu}^2\right)\frac{4}{K - 2N_{U,\mu}} \\
&\leq \left(X_0^2\widetilde{X}_0^2 + 4Y_0\widetilde{Y}_0 + 2N_{U,\mu} + N_\mu\right) + \left(X_0\widetilde{X}_0Y_0\widetilde{Y}_0 + 2N_{U,\mu}Y_0\widetilde{Y}_0 + N_{U,\mu}^2\right)\frac{4}{K - 2N_{U,\mu}} \\
&= \left(X_0\widetilde{X}_0 + \frac{4Y_0\widetilde{Y}_0}{K - 2N_{U,\mu}}\right)X_0\widetilde{X}_0 + \frac{4KY_0\widetilde{Y}_0}{K - 2N_{U,\mu}} + \frac{2KN_{U,\mu}}{K - 2N_{U,\mu}} + N_\mu \\
&< K\left(X_0\widetilde{X}_0 + \frac{4Y_0\widetilde{Y}_0}{K - 2N_{U,\mu}}\right) + \frac{2KN_{U,\mu}}{K - 2N_{U,\mu}} + \frac{KN_\mu}{K - 2N_{U,\mu}} \\
&< K^2 + K(1-K).
\end{aligned}$$

This shows $Z_1 < K$. We now suppose that $Z_j - K \leq K^{j-1}(Z_1 - K) < 0$ for some $j \geq 1$. Then

$$\begin{aligned} Z_{j+1} &\leq Z_j X_j^2 + \frac{4KY_j\widetilde{Y}_j}{K - 2N_{U,\mu}} + K(1-K) \\ &< K\left(X_j^2 + \frac{4Y_j\widetilde{Y}_j}{K - 2N_{U,\mu}}\right) + K(1-K) \\ &\leq K + K(Z_j - K) \\ &\leq K + K^j(Z_1 - K). \end{aligned}$$

This proves the result. □

5.2 A better bound for large j

Note that a simple consequence of Lemma 5.2 is that X_j^2 and $\widetilde{X}_j^2$ are bounded by K and that Y_j^2 and $\widetilde{Y}_j^2$ are bounded by $K(K - 2N_{U,\mu})/4$. In this section we show that for large enough j the improved bound (53) holds. In what follows, write

$$x_j = \max\{X_j^2, \widetilde{X}_j^2\} - (K - 2N_{U,\mu}), \qquad y_j = \frac{4\max\{Y_j^2, \widetilde{Y}_j^2\}}{K - 2N_{U,\mu}} - 2N_{U,\mu}.$$

From the definition of Z_j, we immediately have $x_j + y_j = Z_j - K$, which is negative by Lemma 5.2. In order to prove (53) we must show x_j and y_j are both negative for large enough j.

Proposition 5.3 *Suppose that* (51) *holds. There exists J_X so that $x_j < 0$ for all $j > J_X$.*

Proof: Suppose for a contradiction that $X_j^2 \geq K - 2N_{U,\mu}$ for all j. Now

$$\begin{aligned} Z_{j+1} - K &\leq Z_j X_j^2 + \frac{4KY_j\widetilde{Y}_j}{K - 2N_{U,\mu}} - K^2 \\ &\leq Z_j X_j^2 + K(Z_j - X_j^2) - K^2 \\ &= (K + X_j^2)(Z_j - K) \\ &\leq (2K - 2N_{U,\mu})(Z_j - K) \\ &\leq (2K - 2N_{U,\mu})^j(Z_1 - K). \end{aligned}$$

In the penultimate line we used the hypothesis $X_j^2 \geq K - 2N_{U,\mu}$. Since $K > (1 + 2N_{U,\mu})/2$ we have $(2K - 2N_{U,\mu}) > 1$. Together with $(Z_1 - K) < 0$, this implies there exists $J > 0$ so that

$$Z_{J+1} - K \leq (2K - 2N_{U,\mu})^J(Z_1 - K) < -K.$$

Thus $Z_{J+1} < 0$ which is a contradiction. Hence, there exists $J = J_X$ so that $X_{J_X}^2 < K - 2N_{U,\mu}$.

Suppose now that $X_j^2 < K - 2N_{U,\mu}$. Then

$$\begin{aligned} x_{j+1} &= \max\{X_{j+1}^2, \widetilde{X}_{j+1}^2\} - (K - 2N_{U,\mu}) \\ &\leq X_j^2\widetilde{X}_j^2 + 4Y_j\widetilde{Y}_j - K(K - 2N_{U,\mu}) \\ &< (K - 2N_{U,\mu})\widetilde{X}_j^2 + 4Y_j\widetilde{Y}_j - K(K - 2N_{U,\mu}) \\ &\leq (K - 2N_{U,\mu})(Z_j - K). \end{aligned}$$

Hence $x_{j+1} < 0$. In particular, $X_{j+1}^2 < K - 2N_{U,\mu}$ and so our result follows by induction. □

Proposition 5.4 *If $y_j \geq 0$ then $y_{j+1} \leq (K - 2N_{U,\mu})y_j + 2N_{U,\mu}(Z_j - K)$.*

Proof: Since $y_j \geq 0$ we have $x_j = Z_j - K - y_j < 0$. In particular, $x_j y_j \leq 0$. We have

$$\begin{aligned}
y_{j+1} &= \frac{\max\{4Y_{j+1}^2, 4\widetilde{Y}_{j+1}^2\}}{K - 2N_{U,\mu}} - 2N_{U,\mu} \\
&\leq X_j^2 \frac{4\widetilde{Y}_j^2}{K - 2N_{U,\mu}} + 2N_{U,\mu} \frac{4Y_j\widetilde{Y}_j}{K - 2N_{U,\mu}} + 2N_{U,\mu}(1 - K) - 2N_{U,\mu} \\
&\leq (x_j + K - 2N_{U,\mu})(y_j + 2N_{U,\mu}) + 2N_{U,\mu}(y_j + 2N_{U,\mu}) - 2KN_{U,\mu} \\
&= x_j y_j + 2N_{U,\mu}x_j + Ky_j \\
&\leq (K - 2N_{U,\mu})y_j + 2N_{U,\mu}(Z_j - K),
\end{aligned}$$

where we used $x_j + y_j = Z_j - K$ and $x_j y_j \leq 0$ on the last line. Thus, we obtain the result. □

By iterating the above inequalities we obtain:

Corollary 5.5 *If $y_k \geq 0$ for all $k \leq j$ then*

$$y_j \leq (K - 2N_{U,\mu})^{j-1} y_1 + \big(K^{j-1} - (K - 2N_{U,\mu})^{j-1}\big)(Z_1 - K). \tag{56}$$

Proof: The statement (56) is trivial when $j = 1$. Assume that (56) holds for some $j \geq 1$. Since we know $(Z_j - K) \leq K^{j-1}(Z_1 - K)$ we have

$$\begin{aligned}
y_{j+1} &\leq (K - 2N_{U,\mu})y_j + 2N_{U,\mu}(Z_j - K) \\
&\leq (K - 2N_{U,\mu})\Big((K - 2N_{U,\mu})^{j-1}y_1 + \big(K^{j-1} - (K - 2N_{U,\mu})^{j-1}\big)(Z_1 - K)\Big) \\
&\quad + 2N_{U,\mu}K^{j-1}(Z_1 - K) \\
&= (K - 2N_{U,\mu})^j y_1 + \big(K^j - (K - 2N_{U,\mu})^j\big)(Z_1 - K).
\end{aligned}$$

Then (56) holds for $j + 1$. The result follows by induction. □

We can now show that y_j is eventually negative.

Proposition 5.6 *Suppose that (51) holds and $N_{U,\mu} > 0$. There exists J_Y so that $y_j < 0$ for all $j \geq J_Y$.*

Proof: Assume for a contradiction that $y_j \geq 0$ for all j. Since $K > K - 2N_{U,\mu} > 0$, we can find $J = J_Y$ so that

$$\left(\frac{K}{K - 2N_{U,\mu}}\right)^J > 1 + \frac{y_1}{K - Z_1} \geq 1.$$

This means that

$$y_{J+1} \leq (K - 2N_{U,\mu})^J y_1 + \big(K^J - (K - 2N_{U,\mu})^J\big)(Z_1 - K) < 0.$$

This contradicts our hypothesis that $y_j > 0$ for all j.

Now suppose that $y_j < 0$ for some j. We claim that $y_{j+1} < 0$. If $x_j \geq 0$ then, arguing as in the proof of Proposition 5.4,

$$y_{j+1} \leq x_j y_j + (K - 2N_{U,\mu})y_j + 2N_{U,\mu}(Z_j - K) \leq (K - 2N_{U,\mu})y_j + 2N_{U,\mu}(Z_j - K) < 0.$$

On the other hand, if $x_j < 0$, using $y_j \geq -2N_{U,\mu}$, arguing similarly gives

$$y_{j+1} \leq x_j(y_j + 2N_{U,\mu}) + Ky_j \leq Ky_j < 0.$$

In either case, $y_j < 0$ implies $y_{j+1} < 0$. □

Combining Propositions 5.3 and 5.6 proves (53).

5.3 Convergence of X_j, $\widetilde{X}_j$, Y_j, $\widetilde{Y}_j$

For $j \geq 1$ define

$$M_j = \max\{X_j^2,\, \widetilde{X}_j^2,\, 2Y_j^2/N_{U,\mu},\, 2\widetilde{Y}_j^2/N_{U,\mu}\}. \tag{57}$$

Note that (53) implies that $M_j < K - 2N_{U,\mu}$ for $j > \max\{J_X, J_Y\}$. Moreover, (49) and (50) imply

$$M_{j+1} \leq M_j^2 + 2N_{U,\mu}M_j + (K - 2N_{U,\mu})(1 - K). \tag{58}$$

Proposition 5.7 *Suppose that* (51) *holds. If* $j \geq \max\{J_X, J_Y\}$ *then*

$$M_{j+1} \leq \max\{1 - K,\, M_j\}.$$

Furthermore, for all $\varepsilon > 0$ *there exists* J_ε *so that* $M_j \leq 1 - K + \varepsilon$ *for all* $j \geq J_\varepsilon$*.*

Proof: Consider some $j > \max\{J_X, J_Y\}$. Therefore $M_j < K - 2N_{U,\mu}$.
Suppose first $M_j \leq 1 - K$ then using (58)

$$\begin{aligned} M_{j+1} &\leq M_j^2 + 2N_{U,\mu}M_j + (K - 2N_{U,\mu})(1 - K) \\ &\leq (1 - K)^2 + 2N_{U,\mu}(1 - K) + (K - 2N_{U,\mu})(1 - K) \\ &= 1 - K. \end{aligned}$$

Hence if $M_J \leq 1 - K$ for some J then $M_j \leq 1 - K$ for all $j \geq J$. This proves both parts of the result in this case.

Now suppose $M_j > 1 - K$ then

$$\begin{aligned} M_{j+1} &\leq M_j^2 + 2N_{U,\mu}M_j + (K - 2N_{U,\mu})(1 - K) \\ &= \big(M_j - (K - 2N_{U,\mu})\big)\big(M_j - (1 - K)\big) + M_j \\ &< M_j. \end{aligned}$$

This proves the first part of the result in this case. For the second part, suppose $M_J > 1 - K$ for some $J \geq \max\{J_X, J_Y\}$. Then $M_j < M_J$ for all $j \geq J$. Therefore, for $j \geq J$

$$\begin{aligned} M_{j+1} - (1 - K) &\leq \big(M_j - (K - 2N_{U,\mu}) + 1\big)\big(M_j - (1 - K)\big) \\ &\leq \big(M_J - (K - 2N_{U,\mu}) + 1\big)\big(M_j - (1 - K)\big) \\ &\leq \big(M_J - (K - 2N_{U,\mu}) + 1\big)^{j-J+1}\big(M_J - (1 - K)\big). \end{aligned}$$

Since $J \geq \max\{J_X, J_Y\}$, we have $M_J - (K - 2N_{U,\mu}) + 1 < 1$. Hence, we can find J_ε so that

$$\big(M_J - (K - 2N_{U,\mu}) + 1\big)^{j-J+1}\big(M_J - (1 - K)\big) < \varepsilon$$

for all $j \geq J_\varepsilon$. This proves the second part when $M_j > 1 - K$. □

This completes the proof of Theorem 5.1.

6 Convergence of S_j to T

We are now ready to prove that the S_j converge to T as j tends to infinity under the condition (51). Observing that none of the S_j fix ∞ we see that they are all distinct and so the group $\langle S, T\rangle$ is not discrete. We show this convergence by showing convergence of the nine entries of S_j to the corresponding entries of T. We divide our proof into subsections, each containing convergence of certain entries. The main steps are:

- We will first show that c_j tends to zero as j tends to infinity (Proposition 6.2).
- After showing $\|\alpha_j c_j^{-1/2}\|$, $\|\delta_j \overline{c}_j^{-1/2}\|$ are bounded (Lemma 6.3), we can show that α_j and δ_j both tend to $0 \in \mathbb{H}^{n-1}$ as j tends to infinity (Proposition 6.4).
- We then show the remaining matrix entries are bounded (Lemmas 6.6, 6.7 and Corollaries 6.8, 6.9).
- Using the results obtained so far, we can show that a_j and d_j both tend to μ and A_j tends to U as j tends to infinity (Propositions 6.10 and 6.11).
- Finally, we show that β_j, γ_j and b_j tend to $\sqrt{2}\tau\mu$, $-\sqrt{2}\overline{\mu}\tau$ and $(-\|\tau\|^2+t)\mu$ respectively as j tends to infinity (Propositions 6.12 and 6.13).

Throughout this proof we use the fact that (52) holds for large enough j. We will repeatedly use the following elementary lemma to show certain entries are bounded and others converge.

Lemma 6.1 *Let λ_1, λ_2, D be positive real constants with $\lambda_i < 1$ and $\lambda_1 \neq \lambda_2$. Let $C_j \in \mathbb{R}^+$ be defined iteratively.*

(i) If $C_{j+1} \leq \lambda_1 C_j + D$ for $j \geq 0$ then

$$C_j \leq D/(1-\lambda_1) + \lambda_1^j\Big(C_0 - D/(1-\lambda_1)\Big).$$

In particular, given $\varepsilon > 0$ there exists J_ε so that for all $j \geq J_\varepsilon$

$$C_j \leq D/(1-\lambda_1) + \varepsilon.$$

(ii) If $C_{j+1} \leq \lambda_1 C_j + \lambda_2^j D$ for $j \geq 0$ then

$$C_j \leq \lambda_1^j C_0 + D(\lambda_2^j - \lambda_1^j)/(\lambda_2 - \lambda_1).$$

In particular, $C_j \leq C_0\lambda_1^j + \max\{\lambda_1^j, \lambda_2^j\}D/|\lambda_1 - \lambda_2|$.

6.1 Convergence of c_j

The easiest case is to show that c_j tends to zero. Geometrically, this means that the isometric spheres of S_j have radii tending to infinity as j tends to infinity.

Proposition 6.2 *Suppose that* (51) *holds. Then c_j tends to zero as j tends to infinity.*

Proof: Since $K > 1/2$ we can choose ε so that $0 < \varepsilon < K - 1/2$. Then there exists J_ε so that $X_j^2 < (1-K) + \varepsilon < 1/2$ for all $j \geq J_\varepsilon$. From (40) and (52) for $j \geq J_\varepsilon$ we have

$$|c_{j+1}| = X_j^2|c_j| < |c_j|/2 < \cdots < |c_{J_\varepsilon}|/2^{j-J_\varepsilon+1}.$$

Thus that c_j tends to zero as j tends to infinity. $\square$

6.2 Convergence of α_j and δ_j

In this section we show that α_j and δ_j both tend to the zero vector as j tends to infinity. To do so, we first show their norms are bounded by a constant multiple of $|c_j|^{1/2}$.

Lemma 6.3 *Suppose that* (51) *holds. For any $\varepsilon > 0$ there exists $J_\varepsilon > 0$ so that*

$$\|\alpha_j c_j^{-1/2}\| < \frac{\sqrt{2}}{1-\sqrt{1-K}} + \varepsilon, \quad \|\delta_j \bar{c}_j^{-1/2}\| < \frac{\sqrt{2}}{1-\sqrt{1-K}} + \varepsilon.$$

Proof: Given $\varepsilon_1 > 0$ we know from (52) that there exists J_1 so that for $j \geq J_1$

$$X_j^2 \leq (1-K) + \varepsilon_1.$$

Observe that $\alpha_j c_j^{-1/2} = \sqrt{2}\omega_j c_j^{1/2}$. Therefore equation (36) implies that for $j \geq J_1$

$$\begin{aligned}
\|\alpha_{j+1} c_{j+1}^{-1/2}\| &= \sqrt{2}\|\omega_{j+1}\|\,|c_{j+1}|^{1/2} \\
&= \sqrt{2}\left\|\omega_j + B_j \xi_j \mu \bar{c}_j c_{j+1}^{-1}\right\|\,|c_{j+1}|^{1/2} \\
&\leq \sqrt{2}\|\omega_j\|\,|c_{j+1}|^{1/2} + \sqrt{2}\|\xi_j\|\,|c_j|\,|c_{j+1}|^{-1/2} \\
&= \|\alpha_j c_j^{-1/2}\|\,|c_j|^{-1/2}|c_{j+1}|^{1/2} + \sqrt{2}\|\xi_j\|\,|c_j|\,|c_{j+1}|^{-1/2} \\
&= X_j \|\alpha_j c_j^{-1/2}\| + \sqrt{2} Y_j X_j^{-1} \\
&\leq \sqrt{1-K+\varepsilon_1}\,\|\alpha_j c_j^{-1/2}\| + \sqrt{2}.
\end{aligned}$$

Therefore, using Lemma 6.1, given $\varepsilon_2 > 0$ we can find $J_2 \geq J_1$ so that for $j \geq J_2$ we have

$$\|\alpha_j c_j^{-1/2}\| \leq \frac{\sqrt{2}}{1-\sqrt{1-K+\varepsilon_1}} + \varepsilon_2.$$

Given any $\varepsilon > 0$ it is possible to find $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that

$$\frac{\sqrt{2}}{1-\sqrt{1-K+\varepsilon_1}} + \varepsilon_2 \leq \frac{\sqrt{2}}{1-\sqrt{1-K}} + \varepsilon.$$

This proves the first part. A similar argument holds for $\|\delta_j \bar{c}_j^{-1/2}\|$. □

Proposition 6.4 *Suppose that* (51) *holds. Then α_j and δ_j both tend to $0 \in \mathbb{H}^{n-1}$ as j tends to infinity.*

Proof: Clearly $\|\alpha_j\| = \|\alpha_j c_j^{-1/2}\|\,|c_j|^{1/2}$ and $\|\delta_j\| = \|\delta_j \bar{c}_j^{-1/2}\|\,|c_j|^{1/2}$. Using Proposition 6.2 and Lemma 6.3 we see that c_j tends to zero and $\|\alpha_j c_j^{-1/2}\|$ and $\|\delta_j \bar{c}_j^{-1/2}\|$ are bounded. Thus α_j and δ_j both tend to $0 \in \mathbb{H}^{n-1}$ as j tends to infinity. □

The following estimate will be useful later.

Corollary 6.5 *Suppose that* (51) *holds. There exists J_0 so that for $j \geq J_0$*

$$2Y_j^2 \|\alpha_j c_j^{-1/2}\|^2 < \frac{2N_{U,\mu}}{(\sqrt{2}-1)^2} < 1.$$

Proof: From (52) we have

$$2Y_j^2 \leq N_{U,\mu}(1-K)+\varepsilon_1,$$

and from Lemma 6.3 we have

$$\|\alpha_j c_j^{-1/2}\|^2 \leq \frac{2}{(1-\sqrt{1-K})^2}+\varepsilon_2.$$

Given $\varepsilon > 0$, combining these inequalities for suitable ε_1, $\varepsilon_2 > 0$, we obtain

$$2Y_j^2\|\alpha_j c_j^{-1/2}\|^2 \leq \frac{2N_{U,\mu}(1-K)}{(1-\sqrt{1-K})^2}+\varepsilon.$$

Since $(1-K) < 1/2$ we can choose $\varepsilon > 0$ so that

$$\frac{2N_{U,\mu}(1-K)}{(1-\sqrt{1-K})^2}+\varepsilon < \frac{2N_{U,\mu}(1/2)}{1-\sqrt{1/2})^2} = \frac{2N_{U,\mu}}{(\sqrt{2}-1)^2}.$$

The final inequality follows since

$$N_{U,\mu} < \frac{3-2\sqrt{2+N_\mu}}{2} \leq \frac{(\sqrt{2}-1)^2}{2}.$$

□

6.3 The remaining matrix entries are bounded

In this section we show that the norms of the remaining matrix entries are bounded. Later, this will enable us to show they converge. We begin by showing $|a_j|$ and $|b_j|$ are bounded.

Lemma 6.6 *Suppose that* (51) *holds. Given $\varepsilon > 0$ there exists J_ε so that for all $j \geq J_\varepsilon$*

$$|a_j| < 2+\frac{2\sqrt{2N_{U,\mu}}}{\sqrt{2}-1}+\varepsilon, \quad |d_j| < 2+\frac{2\sqrt{2N_{U,\mu}}}{\sqrt{2}-1}+\varepsilon.$$

Proof: We use (37) to obtain

$$\begin{aligned} |a_{j+1}| &= \left|a_j c_j^{-1}c_{j+1}+\bar{c}_j^{-1}\mu\bar{c}_j-\sqrt{2}\,\bar{c}_j^{-1}\alpha_j^*(B_j\xi_j\mu\bar{c}_j)\right| \\ &\leq |a_j|\,|c_{j+1}|\,|c_j|^{-1}+1+\sqrt{2}\|\xi_j\|\,|c_j|^{-1/2}\|\alpha_j c_j^{-1/2}\| \\ &= X_j^2|a_j|+1+\sqrt{2}Y_j\|\alpha_j c_j^{-1/2}\|. \end{aligned}$$

Using (52) and Corollary 6.5, since $1-K<1/2$ we can find J_0 so that for $j \geq J_0$

$$X_j^2 \leq \frac{1}{2}, \quad \sqrt{2}\,Y_j\|\alpha_j c_j^{-1/2}\| \leq \frac{\sqrt{2N_{U,\mu}}}{\sqrt{2}-1}.$$

Thus the bound for $|a_j|$ follows from Lemma 6.1. The bound for $|d_j|$ follows similarly. □

Lemma 6.7 *Suppose that* (51) *holds. Then $|b_j|$ is bounded above as j tends to infinity.*

Proof: If $a_j = 0$ then $\gamma_j = 0$ and so $b_{j+1} = 0$. Hence we take $a_j \neq 0$. Then (11) gives

$$\begin{aligned} 0 &= (a_j\bar{b}_j + \gamma_j^*\gamma_j + b_j\bar{a}_j)\bar{a}_j^{-1}\mu\bar{a}_j = a_j\bar{b}_j\bar{a}_j^{-1}\mu\bar{a}_j + \gamma_j^*\gamma_j\bar{a}_j^{-1}\mu\bar{a}_j + b_j\mu\bar{a}_j, \\ \|\gamma_j\|^2 &= -(a_j\bar{b}_j + b_j\bar{a}_j) \leq 2|a_j|\,|b_j|. \end{aligned}$$

Hence, using (22), we have

$$\begin{aligned} b_{j+1} &= \gamma_j^*U\gamma_j - \sqrt{2}a_j\tau^*\mu\gamma_j + \sqrt{2}\gamma_j^*\tau\mu\bar{a}_j - a_j(\|\tau\|^2 - t)\mu\bar{a}_j + a_j\mu\bar{b}_j + b_j\mu\bar{a}_j \\ &= \gamma_j^*U(\gamma_j\bar{a}_j^{-1})\bar{a}_j - \sqrt{2}a_j\tau^*\mu\gamma_j + \sqrt{2}\gamma_j^*\tau\mu\bar{a}_j - a_j(\|\tau\|^2 - t)\mu\bar{a}_j + a_j\mu(\bar{b}_j\bar{a}_j^{-1})\bar{a}_j + b_j\mu\bar{a}_j \\ &\quad -\gamma_j^*(\gamma_j\bar{a}_j^{-1})\mu\bar{a}_j - a_j(\bar{b}_j\bar{a}_j^{-1})\mu\bar{a}_j - b_j\mu\bar{a}_j \\ &= \gamma_j^*(U\gamma_j\bar{a}_j^{-1} - \gamma_j\bar{a}_j^{-1}\mu)\bar{a}_j - \sqrt{2}a_j\tau^*\mu\gamma_j + \sqrt{2}\gamma_j^*\tau\mu\bar{a}_j - a_j(\|\tau\|^2 - t)\mu\bar{a}_j \\ &\quad +a_j(\mu\bar{b}_j\bar{a}_j^{-1} - \bar{b}_j\bar{a}_j^{-1}\mu)\bar{a}_j. \end{aligned}$$

Then we have

$$\begin{aligned} |b_{j+1}| &\leq |\gamma_j^*(U\gamma_j\bar{a}_j^{-1} - \gamma_j\bar{a}_j^{-1}\mu)\bar{a}_j| + \sqrt{2}|a_j\tau^*\mu\gamma_j| + \sqrt{2}|\gamma_j^*\tau\mu\bar{a}_j| + |a_j(\|\tau\|^2 - t)\mu\bar{a}_j| \\ &\quad +|a_j(\mu\bar{b}_j\bar{a}_j^{-1} - \bar{b}_j\bar{a}_j^{-1}\mu)\bar{a}_j| \\ &\leq N_{U,\mu}\|\gamma_j\|^2 + 2\sqrt{2}|a_j|\,\|\tau\|\,\|\gamma_j\| + |a_j|^2\big|\|\tau\|^2 - t\big| + N_\mu|a_j|\,|b_j| \\ &\leq (2N_{U,\mu} + N_\mu)|a_j|\,|b_j| + 4|a_j|^{3/2}\|\tau\|\,|b_j|^{1/2} + |a_j|^2\big|\|\tau\|^2 - t\big|. \end{aligned}$$

Observe that our hypotheses $N_\mu < 1/4$ and $N_{U,\mu} < (3 - 2\sqrt{2 + N_\mu})/2$ imply that

$$2N_{U,\mu} + N_\mu < N_\mu + 3 - 2\sqrt{2 + N_\mu} = \left(\sqrt{2 + N_\mu} - 1\right)^2 \leq (3/2 - 1)^2 = 1/4. \tag{59}$$

Since $\sqrt{2N_{U,\mu}} < \sqrt{2} - 1$, there exists L with

$$\frac{1}{2}\left(1 + \frac{\sqrt{2N_{U,\mu}}}{\sqrt{2} - 1}\right) < L^2 < 1.$$

Hence, using Lemma 6.6, we can find J_L so that for all $j \geq J_L$ we have $|a_j| \leq 4L^2$. In particular,

$$(2N_{U,\mu} + N_\mu)|a_j| \leq L^2.$$

Therefore, for $j \geq J_L$

$$\begin{aligned} |b_{j+1}| &\leq L^2|b_j| + 32L^3\|\tau\|\,|b_j|^{1/2} + 16L^4\big|\,\|\tau\|^2 - t\big| \\ &\leq \left(L|b_j|^{1/2} + 16L^2\big|\|\tau\|^2 - t\big|^{1/2}\right)^2. \end{aligned}$$

In other words

$$|b_{j+1}|^{1/2} \leq L|b_j|^{1/2} + 16L^2\big|\|\tau\|^2 - t\big|^{1/2}.$$

Since $L < 1$ we see that $|b_j|$ is bounded using Lemma 6.1. □

Corollary 6.8 *Suppose that* (51) *holds. Then* $\|\beta_j\|$ *and* $\|\gamma_j\|$ *are bounded above as* j *tends to infinity.*

Proof: Note that $\|\gamma_j\|^2 = -(a_j\bar{b}_j + b_j\bar{a}_j) \le 2|a_j||b_j|$ and $\|\beta_j\|^2 = -(\bar{b}_j d_j + \bar{d}_j b_j) \le 2|b_j||d_j|$. Thus Lemmas 6.6 and 6.7 imply that $\|\beta_j\|$ and $\|\gamma_j\|$ are bounded. □

Finally, we show that $\|A_j\|$ and $\|A_j - U\|$ are bounded.

Corollary 6.9 *Suppose that* (51) *holds. Then* $\|A_j\|$ *and* $\|A_j - U\|$ *are bounded as* j *tends to* ∞.

Proof: Using (13) we have

$$\begin{aligned} I_{n-1} &= A_jA_j^* + \alpha_j\beta_j^* + \beta_j\alpha_j^* \\ &= (A_j - U)(A_j^* - U^*) + U(A_j^* - U^*) + (A_j - U)U^* + I_{n+1} + \alpha_j\beta_j^* + \beta_j\alpha_j^*. \end{aligned}$$

Therefore

$$\|A_j - U\|^2 \le 2\|A_j - U\| + 2\|\alpha_j\|\,\|\beta_j\|,$$

which implies that

$$\|A_j - U\| \le 1 + \sqrt{1 + 2\|\alpha_j\|\,\|\beta_j\|}. \tag{60}$$

Hence $\|A_j - U\|$ and $\|A_j\|$ are bounded. □

6.4 Convergence of a_j and d_j

We now show a_j and d_j both tend to μ.

Proposition 6.10 *Suppose that* (51) *holds. Then both* a_j *and* d_j *tend to* μ *as* j *tends to infinity.*

Proof: Recall from (10) that $1 = a_j\bar{d}_j + \gamma_j^*\delta_j + b_j\bar{c}_j$. Using (20), we have

$$\begin{aligned} a_{j+1} - \mu &= \gamma_j^*U\delta_j - \sqrt{2}a_j\tau^*\mu\delta_j + \sqrt{2}\gamma_j^*\tau\mu\bar{c}_j - a_j\big(\|\tau\|^2 - t\big)\mu\bar{c}_j + a_j\mu\bar{d}_j + b_j\mu\bar{c}_j \\ &\quad -\mu\gamma_j^*\delta_j - \mu a_j\bar{d}_j - \mu b_j\bar{c}_j \\ &= (\gamma_j^*U - \mu\gamma_j^*)\delta_j - \sqrt{2}a_j\tau^*\mu\delta_j + \sqrt{2}\gamma_j^*\tau\mu\bar{c}_j - a_j\big(\|\tau\|^2 - t\big)\mu\bar{c}_j \\ &\quad +\big((a_j - \mu)\mu - \mu(a_j - \mu)\big)\bar{d}_j + (b_j\mu - \mu b_j)\bar{c}_j. \end{aligned}$$

Hence

$$\begin{aligned} |a_{j+1} - \mu| &\le N_{U,\mu}\|\gamma_j\|\,\|\delta_j\| + \sqrt{2}\|\tau\|\,|a_j|\,\|\delta_j\| + \sqrt{2}\|\tau\|\,|c_j|\,\|\gamma_j\| + \big|\|\tau\|^2 - t\big|\,|a_j|\,|c_j| \\ &\quad +N_\mu|d_j|\,|a_j - \mu| + N_\mu|b_j|\,|c_j| \\ &\le N_\mu|d_j|\,|a_j - \mu| + \Big(N_{U,\mu}\|\gamma_j\| + \sqrt{2}\|\tau\|\,|a_j|\Big)\|\delta_j\bar{c}_j^{-1/2}\|\,|c_j|^{1/2} \\ &\quad +\Big(\sqrt{2}\|\tau\|\,\|\gamma_j\| + \big|\|\tau\|^2 - t\big|\,|a_j| + N_\mu|b_j|\Big)|c_j|. \end{aligned}$$

Suppose that ε satisfies

$$0 < \varepsilon < 1 - \frac{\sqrt{2N_{U,\mu}}}{\sqrt{2} - 1}.$$

Then, using Lemma 6.6, we can find J_1 so that for $j \ge J_1$

$$|d_j| < 2 + \frac{2\sqrt{2N_{U,\mu}}}{\sqrt{2} - 1} + 2\varepsilon < 4.$$

Using $N_\mu < 1/4$ this means that for $j \geq J_\varepsilon$ we have $N_\mu|d_j| < 1$. Moreover, for $j \geq J_1$ we have $X_k^2 \leq 1/2$. Therefore $|c_j| \leq |c_{J_1}|/2^{j-J_1}$. Then using Lemma 6.1 with $\lambda_1 = N_\mu|d_j| < 1$ and $\lambda_2 = |c_j|^{1/2} \leq 1/\sqrt{2}$ we see that $|a_j - \mu|$ tends to 0 as j tends to infinity.

Similarly $|d_j - \mu|$ tends to zero as j tends to infinity. □

6.5 Convergence of A_j

We now show that A_j tends to U.

Proposition 6.11 *Suppose that* (51) *holds. Then A_j tends to U as j tends to infinity.*

Proof: Recall from Corollary 6.9 that $\|A_j\|$ and $\|A_j - U\|$ are bounded. Note that

$$A_jU - UA_j = \big((A_j - U)U - \mu(A_j - U)\big) + \big(\mu(A_j - U) - (A_j - U)\mu\big) - \Big(U(A_j - U) - (A_j - U)\mu\Big).$$

Therefore

$$\|A_jU - UA_j\| \leq (2N_{U,\mu} + N_\mu)\|A_j - U\|.$$

Hence

$$\begin{aligned} \|A_jUA_j^* - UA_jA_j^*\| &= \big\|(A_jU - UA_j)(A^* - U^*) + (A_jU - UA_j)U^*\big\| \\ &\leq \|A_jU - UA_j\|\big(\|A_j - U\| + 1\big) \\ &\leq (2N_{U,\mu} + N_\mu)\|A_j - U\|\big(\|A_j - U\| + 1\big). \end{aligned}$$

From (60) we have

$$(2N_{U,\mu} + N_\mu)\big(\|A_j - U\| + 1\big) \leq (2N_{U,\mu} + N_\mu)\left(2 + \sqrt{1 + 2\|\alpha_j\|\,\|\beta_j\|}\right).$$

Since $2N_{U,\mu} + N_\mu < 1/4$ by (59), $\|\beta_j\|$ is bounded and $\|\alpha_j\|$ tends to zero, we can find J so that for all $j \geq J$ we have

$$\|A_jUA_j^* - UA_jA_j^*\| < \frac{2+\sqrt{2}}{4}\|A_j - U\|.$$

Noting that $U = U\alpha_j\beta_j^* + UA_jA_j^* + U\beta_j\alpha_j^*$, we use (24) to find that

$$\begin{aligned} A_{j+1} - U &= A_jUA_j^* - \sqrt{2}\alpha_j\tau^*\mu A_j^* + \sqrt{2}A_j\tau\mu\alpha_j^* - \alpha_j(\|\tau\|^2 - t)\mu\alpha_j^* + \alpha_j\mu\beta_j^* + \beta_j\mu\alpha_j^* \\ &\quad - UA_jA_j^* - U\alpha_j\beta_j^* - U\beta_j\alpha_j^* \\ &= A_jUA_j^* - UA_jA_j^* - \sqrt{2}\alpha_j\tau^*\mu(A_j^* - U^*) + \sqrt{2}(A_j - U)\tau\mu\alpha_j^* - \alpha_j(\|\tau\|^2 - t)\mu\alpha_j^* \\ &\quad - \sqrt{2}\alpha_j\tau^* + \sqrt{2}U\tau\mu\alpha_j^* - (U\alpha_j - \alpha_j\mu)\beta_j^* - (U\beta_j - \beta_j\mu)\alpha_j^*. \end{aligned}$$

Note, we have used $\tau^*U = \tau^*\mu$. Thus for $j \geq J$,

$$\begin{aligned} \|A_{j+1} - U\| &\leq \|A_jUA_j^* - UA_jA_j^*\| + 2\sqrt{2}\|A_j - U\|\,\|\alpha_j\|\,\|\tau\| + \big|\|\tau\|^2 - t\big|\,\|\alpha_j\|^2 \\ &\quad + 2\sqrt{2}\|\tau\|\,\|\alpha_j\| + 2N_{U,\mu}\|\alpha_j\|\,\|\beta_j\| \\ &< \frac{2+\sqrt{2}}{4}\|A_j - U\| + \big|\|\tau\|^2 - t\big|\,\|\alpha_jc_j^{-1/2}\|^2|c_j| \\ &\quad + \Big(2\sqrt{2}\|A_j - U\|\,\|\tau\| + 2\sqrt{2}\|\tau\| + 2N_{U,\mu}\|\beta_j\|\Big)\|\alpha_jc_j^{-1/2}\|\,|c_j|^{1/2}. \end{aligned}$$

Suppose that J is large enough that for $j \geq J$ we have $|c_j| \leq |c_J|/2^{j-J}$. Now apply Lemma 6.1 with $\lambda_1 = (2+\sqrt{2})/4$ and $\lambda_2 = 1/\sqrt{2}$, and so $\|A_j - U\|$ tends to zero as j tends to infinity. □

6.6 Convergence of β_j and γ_j

We are now ready to show convergence of β_j and γ_j.

Proposition 6.12 *Suppose that* (51) *holds. Then* β_j, *and* γ_j *tend to* $\sqrt{2}\tau\mu$ *and* $-\sqrt{2}\overline{\mu}\tau$ *respectively as* j *tends to infinity.*

Proof: Using $U\beta_j\overline{a}_j + UA_j\gamma_j + U\alpha_j\overline{b}_j = 0$, which follows from (14), we have

$$\begin{aligned}
&\beta_{j+1} - \sqrt{2}\tau\mu \\
&\quad= A_jU\gamma_j - \sqrt{2}\alpha_j\tau^*\mu\gamma_j + \sqrt{2}A_j\tau\mu\overline{a}_j - \alpha_j(\|\tau\|^2 - t)\mu\overline{a}_j + \alpha_j\mu\overline{b}_j + \beta_j\mu\overline{a}_j - \sqrt{2}\tau\mu \\
&\quad= A_jU\gamma_j - \sqrt{2}\alpha_j\tau^*\mu\gamma_j + \sqrt{2}A_j\tau\mu\overline{a}_j - \alpha_j(\|\tau\|^2 - t)\mu\overline{a}_j + \alpha_j\mu\overline{b}_j + \beta_j\mu\overline{a}_j - \sqrt{2}\tau\mu \\
&\qquad -UA_j\gamma_j - U\alpha_j\overline{b}_j - U\beta_j\overline{a}_j \\
&\quad= (A_jU - UA_j)\gamma_j - \sqrt{2}\alpha_j\tau^*\mu\gamma_j - \alpha_j(\|\tau\|^2 - t)\mu\overline{a}_j - (U\alpha_j - \alpha_j\mu)\overline{b}_j \\
&\qquad +\sqrt{2}(A_j - U)\tau\mu\overline{a}_j - \big(U(\beta_j - \sqrt{2}\tau\mu) - (\beta_j - \sqrt{2}\tau\mu)\mu\big)\overline{a}_j + \sqrt{2}\tau\mu^2(\overline{a}_j - \overline{\mu})
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\beta_{j+1} - \sqrt{2}\tau\mu\| \;\leq\; & N_{U,\mu}|a_j|\,\|\beta_j - \sqrt{2}\tau\mu\| + \Big(2\|\gamma_j\| + \sqrt{2}\|\tau\|\,|a_j|\Big)\|A_j - U\| \\
& + \Big(\sqrt{2}\|\tau\|\,\|\gamma_j\| + \big|\|\tau\|^2 - t\big|\,|a_j| + N_{U,\mu}|b_j|\Big)\|\alpha_j c_j^{-1/2}\|\,|c_j|^{1/2}.
\end{aligned}$$

Suppose that j is large enough that $N_{U,\mu}|a_j| < 1$ (compare the proof of Lemma 6.7). Since $|c_j|^{1/2}$ and $\|A_j - U\|$ are bounded by a constant multiple of $2^{(j-J)/2}$ we can apply Lemma 6.1 to show that $\|\beta_j - \sqrt{2}\tau\mu\|$ tends to zero as j tends to infinity. A similar argument shows that $\|\gamma_j + \sqrt{2}\overline{\mu}\tau\|$ tends to zero as j tends to infinity. This argument uses $U^*\tau = \overline{\mu}\tau$. □

6.7 Convergence of b_j

Finally, we show that b_j converges as j tends to infinity.

Proposition 6.13 *Suppose that* (51) *holds. Then* b_j *tends to* $-\big(\|\tau\|^2 - t\big)\mu$ *as* j *tends to infinity.*

Proof: Note that if b_j tends to $-\big(\|\tau\|^2 - t\big)\mu$ then $\overline{b}_j$ tends to $-\overline{\mu}\big(\|\tau\|^2 + t\big)$.
Using $0 = \gamma_j^*\gamma_j\mu + a_j\overline{b}_j\mu + b_j\overline{a}_j\mu$, we have

$$\begin{aligned}
&b_{j+1} + \big(\|\tau\|^2 - t\big)\mu \\
&\quad= \gamma_j^*U\gamma_j - \sqrt{2}a_j\tau^*\mu\gamma_j + \sqrt{2}\gamma_j^*\tau\mu\overline{a}_j - a_j\big(\|\tau\|^2 - t\big)\mu\overline{a}_j + a_j\mu\overline{b}_j + b_j\mu\overline{a}_j \\
&\qquad -\gamma_j^*\gamma_j\mu - a_j\overline{b}_j\mu - b_j\overline{a}_j\mu + \big(\|\tau\|^2 - t\big)\mu \\
&\quad= \gamma_j^*U(\gamma_j + \sqrt{2}\overline{\mu}\tau) - \gamma_j^*(\gamma_j + \sqrt{2}\overline{\mu}\tau)\mu + \sqrt{2}(\gamma_j^* + \sqrt{2}\tau^*\mu)\overline{\mu}\tau\mu - 2\|\tau\|^2\mu \\
&\qquad -\sqrt{2}a_j\tau^*\mu(\gamma_j + \sqrt{2}\overline{\mu}\tau) + 2a_j\|\tau\|^2 + \sqrt{2}\gamma_j^*\tau\mu(\overline{a}_j - \overline{\mu}) \\
&\qquad -a_j\big(\|\tau\|^2 - t\big)\mu(\overline{a}_j - \overline{\mu}) - a_j\big(\|\tau\|^2 - t\big) + a_j\mu\big(\overline{b}_j + \overline{\mu}\big(\|\tau\|^2 + t\big)\big) - a_j\big(\|\tau\|^2 + t\big) \\
&\qquad +b_j\mu(\overline{a}_j - \overline{\mu}) - a_j\big(\overline{b}_j + \overline{\mu}\big(\|\tau\|^2 + t\big)\big)\mu + a_j\overline{\mu}\big(\|\tau\|^2 + t\big)\mu - b_j(\overline{a}_j - \overline{\mu})\mu + \big(\|\tau\|^2 - t\big)\mu \\
&\quad= \gamma_j^*\big(U(\gamma_j + \sqrt{2}\overline{\mu}\tau) - (\gamma_j + \sqrt{2}\overline{\mu}\tau)\mu\big) + \sqrt{2}(\gamma_j^* + \sqrt{2}\tau^*\mu)\overline{\mu}\tau\mu - \sqrt{2}a_j\tau^*\mu(\gamma_j + \sqrt{2}\overline{\mu}\tau) \\
&\qquad +\sqrt{2}\gamma_j^*\tau\mu(\overline{a}_j - \overline{\mu}) - a_j\big(\|\tau\|^2 - t\big)\mu(\overline{a}_j - \overline{\mu}) + b_j\big(\mu(\overline{a}_j - \overline{\mu}) - (\overline{a}_j - \overline{\mu})\mu\big) \\
&\qquad +(a_j - \mu)\overline{\mu}\big(\|\tau\|^2 + t\big)\mu + a_j\big(\mu\big(\overline{b}_j + \overline{\mu}\big(\|\tau\|^2 + t\big)\big) - \big(\overline{b}_j + \overline{\mu}\big(\|\tau\|^2 + t\big)\big)\mu\big).
\end{aligned}$$

Therefore

$$\begin{aligned}\big|b_{j+1}+\big(\|\tau\|^2-t\big)\mu\big| \quad\leq\quad & \big(N_{U,\mu}\|\gamma_j\|+\sqrt{2}\|\tau\|(|a_j|+1)\big)\big\|\gamma_j+\sqrt{2}\overline{\mu}\tau\big\| \\ & +\big(\sqrt{2}\|\gamma_j\|\,\|\tau\|+\big|\|\tau\|^2-t\big|(|a_j|+1)+N_\mu|b_j|\big)|a_j-\mu| \\ & +N_\mu|a_j|\,\big|b_j+\big(\|\tau\|^2-t\big)\mu\big|.\end{aligned}$$

We can take j large enough that $N_\mu|a_j|<1$. Also, we know that $\big\|\gamma_j+\sqrt{2}\overline{\mu}\tau\big\|$ and $|a_j-\mu|$ are bounded by constant multiples of $2^{(j-J)/2}$. Therefore, we can apply Lemma 6.1 to conclude that $\big|b_j+\big(\|\tau\|^2-t\big)\mu\big|$ tends to zero. □

Propositions 6.2 to 6.13 imply that S_j tends to T as j tends to infinity, which completes the proof of Theorem 1.1.

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