Conformal random growth models
Lecture 2: Scaling limits

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Hastings-Levitov model

Let $\theta_1, \theta_2, \cdots$ be i.i.d. uniformly distributed on $[0, 2\pi)$.
For a particle map $f : \Delta \to \Delta \setminus P$ as defined previously with $f(z) = e^{\mathbf{c}} z + O(1)$ near ∞, define the rotated map

$$f_n(w) = e^{i\theta_n} f(e^{-i\theta_n} w).$$

Let $\Phi_n = f_1 \circ \cdots \circ f_n$, then $\mathbb{C} \setminus \Phi_n(\Delta) =: K_n$ is the Hastings-Levitov cluster with n particles, each of capacity $\mathbf{c}$.

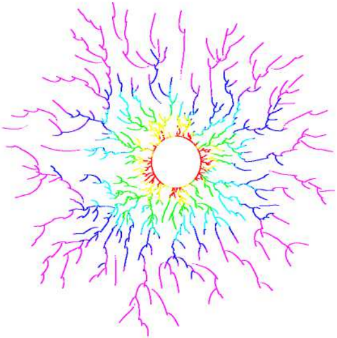
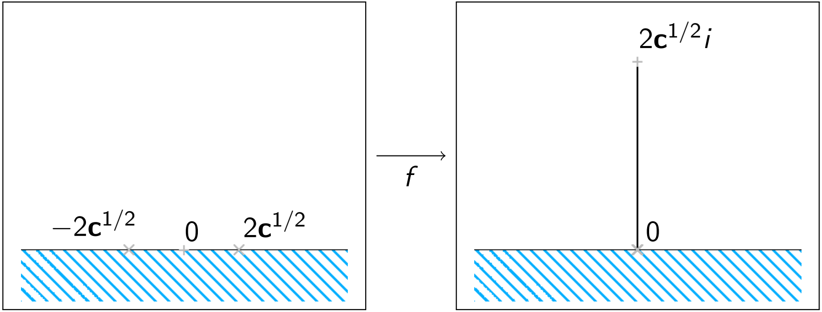


Figure: A simulation of the HL(0) process, taken from *Norris and Turner 2012*.

An example of a particle map

One particle we may attach is a slit of length d.
We have an explicit formula for $f(w)$ here, but for simplicity let's look at the half-plane version.

$$f : \mathbb{H} \to \mathbb{H} \setminus (0, 2\mathbf{c}^{1/2} i],$$
$$f(w) = \sqrt{z^2 - 4\mathbf{c}}.$$



"Small particle limit" - in what space do we converge?

To get a scaling limit, we allow the logarithmic capacity **c** of our particle to tend to zero.
One reasonable question: when we say "limit", what space does this limit live in, and in what sense can we converge to it?

Definition

Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of domains in $\mathbb{C}_\infty \setminus \{0\}$ whose intersection contains a neighbourhood of ∞.
The *kernel* of the sequence is the largest domain D containing ∞ such that every compact subset of D is a subset of all but finitely many of the D_ns.
If every subsequence of $(D_n)_{n\in\mathbb{N}}$ has the same kernel D, then we say that $D_n \to D$ as $n \to \infty$.

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An example of Carathéodory convergence

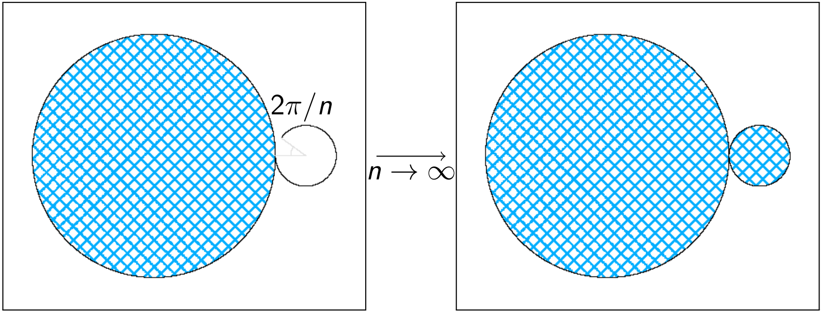

Figure: A diagram of a sequence of sets converging in the Carathéodory sense.

HL(0) scaling limit

Theorem (Norris and Turner, 2012)

Let K_n be the Hastings-Levitov cluster with n particles each of capacity $\mathbf{c}$. In the limit $\mathbf{c} \to 0$ with $n\mathbf{c} \to t$, the cluster K_n converges (in the sense of Carathéodory) to a disc of radius e^t.

This theorem looks daunting to prove. It would be nice if we had something more explicit to work with than the Carathéodory topology.

Let $\mathcal{S}$ be the set of all conformal maps $\varphi\colon \mathbb{D} \to D$ for simply connected domains $D \subsetneq \mathbb{C}$ with $0 \in D$ such that $\varphi(0) = 0$, $\varphi'(0) \in \mathbb{R}_{>0}$.
Note that φ uniquely determines D, and vice versa.

Theorem (Carathéodory, 1912)

Let $\varphi, \varphi_n \in \mathcal{S}$ for $n \geq 1$, and D, D_n the corresponding domains. Then $D_n \to D$ as before if and only if $\varphi_n \to \varphi$ uniformly on compact subsets of Δ.

The Carathéodory convergence theorem

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Convergence of the HL(0) maps

Proposition (Norris and Turner, 2012)

Let K_n be the HL(0) cluster with n particles of capacity $\mathbf{c}$, and $\Phi_n \colon \Delta \to \mathbb{C}_\infty \setminus K_n$ the corresponding map. Again send $\mathbf{c} \to 0$ with $n\mathbf{c} \to t$, then for any compact subset $C \subset \Delta$,

$$\sup_{w \in C} |\Phi_n(w) - e^t w| \to 0$$

in probability.

Two neat tricks

Definition (Logarithmic coordinates)

For $(\Delta \xrightarrow{f} D) \in \mathcal{S}$, let $\widetilde{D} = \{z \in \mathbb{C} : e^z \in D\}$, and $\widetilde{f} : \widetilde{\Delta} \to \widetilde{D}$ the unique conformal map with $\lim_{\Re(w) \to +\infty} (\widetilde{f}(w) - w) = \mathbf{c}$ (where $\mathbf{c}$ is the logarithmic capacity of D^c).

We can also characterise $\widetilde{f}$ by $f \circ \exp = \exp \circ \widetilde{f}$, and so if $f_1, \cdots, f_n$ are the first n particle maps for HL(0) then $\widetilde{\Phi}_n = \widetilde{f}_1 \circ \cdots \circ \widetilde{f}_n$.

Definition (The inverse functions)

We write $g_n = f_n^{-1}$ and $\Gamma_n = \Phi_n^{-1}$.

This is very useful, because for all $z \in K_N^c$ the stochastic process $(\Gamma_n(z) : 0 \leq n \leq N)$ is *Markovian*.

Logarithmic coordinates

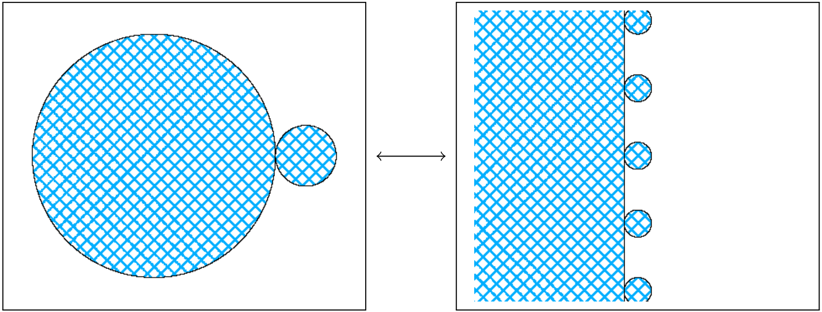

Figure: A drawing of a particle in the usual (left) and logarithmic (right) coordinates.

Convergence of maps

Consider a particular event $\Omega(m, \varepsilon)$ for $m \in \mathbb{N}$ and small ε defined by two conditions: For all $n \leq m$,

- $\left|\widetilde{\Phi}_n(w) - (w + n\mathbf{c})\right| < \varepsilon$ whenever $\Re(w) \geq 5\varepsilon$.
- $z \in \widetilde{D}_n$ and $\left|\widetilde{\Gamma}_n(z) - (z - n\mathbf{c})\right| < \varepsilon$ whenever $\Re(z) \geq n\mathbf{c} + 4\varepsilon$.

We claim that on this event (if $\varepsilon \to 0$ and $m \to \infty$ at appropriate speeds) the cluster converges to a disc of radius e^t.
If $e^w = z$ then

$$|\Phi_n(z) - e^{n\mathbf{c}} z| = |\exp(\widetilde{\Phi}_n(w)) - \exp(w + n\mathbf{c})| < \varepsilon e^{6\varepsilon + n\mathbf{c}}.$$

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The good event is likely

Proposition (Norris and Turner, 2012)

We use a particle P satisfying, for $\delta \leq 1/3$,

$$P \subseteq \{z \in \Delta : |z-1| \leq \delta\}, \quad 1+\delta \in P, \quad z \in P \iff \bar{z} \in P.$$

There is a constant A such that for all $2\delta \leq \varepsilon \leq 1$ and $m \geq 1$ we have

$$\mathbb{P}(\Omega(m,\varepsilon)^c) \leq A(m+\varepsilon^{-2}) \exp\left(-\frac{\varepsilon^3}{A\mathbf{c}}\right)$$

for a constant A.

If $\varepsilon \to 0$ slowly enough as $\mathbf{c} \to 0$, then $\mathbb{P}(\Omega(m,\varepsilon)) \to 1$, and so we get our convergence result.

Proof: outline

Our plan for bounding the probability of the bad event has several steps:

- Write the event $\Omega(m, \varepsilon)$ (which talks about *all* w in a half-plane) as the intersection of events Ω_R depending on vertical lines $\ell_R = \{\zeta \in \mathbb{C} : \Re(\zeta) = R\}$.
- Work only with the Markovian $\widetilde{\Gamma}_n$, and deduce the result for $\widetilde{\Phi}_n$ from the result for $\widetilde{\Gamma}_n$.
- On each event, express the difference $\left|\widetilde{\Gamma}_n(z) - (z - n\mathbf{c})\right|$ as a martingale in n, and bound its size.

Let $R = 2(k+1)\varepsilon$ for some $k \in \mathbb{N}$. Let N be the maximal integer such that $R \geq 2\varepsilon + \mathbf{c}N$. Consider the stopping time

$$T_R = \inf\{n \geq 0 : \text{for some } z \in \ell_R, z \in \widetilde{K}_n \text{ or } \Re(\widetilde{\Gamma}_n(z)) \leq R - n\mathbf{c} - \varepsilon\} \wedge N,$$

and define the event

$$\Omega_R = \left\{ \sup_{n \leq T_R, z \in \ell_R} |\widetilde{\Gamma}_n(z) - (z - n\mathbf{c})| < \varepsilon \right\}.$$

We claim that $\Omega(m, \varepsilon) \supseteq \bigcap_{k=1}^{\lceil m\mathbf{c}/2\varepsilon \rceil} \Omega_{2(k+1)\varepsilon}$ (this is easy thanks to the magic of holomorphicity).

Proof: some technical details

Consider $\widetilde{g}$, the unique conformal map $\widetilde{\Delta \setminus P} \to \widetilde{\Delta}$ with $\lim_{\Re(z)\to+\infty}(\widetilde{g}(z) - z) = -\mathbf{c}$.

We need a few facts about $\widetilde{g}$. Let $\widetilde{g}_0(z) = \widetilde{g}(z) - z$ for convenience. By Cauchy's integral formula, we have whenever $\Re(z) > \delta$,

$$\frac{1}{2\pi}\int_0^{2\pi} \widetilde{g}_0(z - i\theta)\,\mathrm{d}\theta = -\mathbf{c}.$$

From an earlier section in the paper, when $\Re(z) \geq 2\delta$ we also have, writing $q(r) = r \wedge r^2$,

$$|\widetilde{g}_0(z) + \mathbf{c}| \leq \frac{A\mathbf{c}}{\Re(z) - \delta}, \quad |\widetilde{g}_0'(z)| \leq \frac{2A\mathbf{c}}{q(\Re(z) - \delta)},$$

where A is a universal constant.

Proof: a martingale

Fix an $R = 2(k+1)\varepsilon$ with $1 \leq k \leq \lceil m\mathbf{c}/2\varepsilon \rceil$. For $z \in \widetilde{D}_n$, define

$$M_n(z) = \widetilde{\Gamma}_n(z) - (z - n\mathbf{c}).$$

We claim this is a martingale.

Proof.

$$\begin{aligned} M_{n+1}(z) &= \widetilde{\Gamma}_{n+1}(z) + (n+1)\mathbf{c} \\ &= \widetilde{g}(\widetilde{\Gamma}_n(z) - i\theta_{n+1}) + i\theta_{n+1} + (n+1)\mathbf{c} \\ &= \widetilde{g}_0(\widetilde{\Gamma}_n(z) - i\theta_{n+1}) + \widetilde{\Gamma}_n(z) + n\mathbf{c} + \mathbf{c}, \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E}[M_{n+1}(z) - M_n(z) | \theta_1, \cdots, \theta_n] &= \frac{1}{2\pi} \int_0^{2\pi} \widetilde{g}_0(\widetilde{\Gamma}_n(z) - i\theta) \, \mathrm{d}\theta + \mathbf{c} \\ &= 0. \end{aligned}$$

Proof: analysing our martingale

We can bound the increments, so for $z \in \ell_R$ and $n < T_R$, using our earlier technical estimate,

$$|M_{n+1}(z) - M_n(z)| \leq \frac{A\mathbf{c}}{\Re(\widetilde{\Gamma}_n(z)) - \delta} \leq \frac{A\mathbf{c}}{R - n\mathbf{c} - \varepsilon - \delta}.$$

Using a martingale gives us lots of useful tools.

Theorem (Azuma-Hoeffding inequality)

Let $(X_n)_{n\geq 0}$ be a martingale with $X_0 = 0$, and $(x_n)_{n\geq 0}$ a sequence of positive reals such that $|X_{n+1} - X_n| \leq x_n$ for all n. Then for $\lambda > 0$,

$$\mathbb{P}(|X_n| \geq \lambda) \leq 2\exp\left(\frac{-\lambda^2}{2\sum_{k=1}^{n-1} x_k^2}\right)$$

Applying this, since $\sum_{n=1}^{N-1}\left(\frac{A\mathbf{c}}{R-n\mathbf{c}-\varepsilon-\delta}\right)^2 \leq \frac{2A^2\mathbf{c}}{\varepsilon}$, we have

$$\mathbb{P}\left(\sup_{n<T} |M_n(z)| \geq \varepsilon/2\right) \leq 2\exp\left(\frac{-\varepsilon^3}{16A^2\mathbf{c}}\right) \quad (1)$$

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Proof: From a pointwise to a global estimate

We have a family of martingales indexed by $z \in \ell_R$. We have a pointwise bound on each martingale, but can we bound the supremum on ℓ_R?
For $z, z' \in \ell_R$ let $I_n = M_n(z) - M_n(z')$. Consider the function

$$s(n) = \mathbb{E}\left(\sup_{k \leq T_R \wedge n} |I_k|^2\right).$$

If we can bound $s(N)$ by something in terms of $|z - z'|$ then Kolmogorov's continuity theorem allows us to bound $|I_n|$ in terms of $M|z - z'|^\gamma$ for some $\gamma > 0$ and an r.v. M.

Another useful martingale result is Doob's L^2 inequality:

$$\mathbb{E}(|I_n|^2) \leq \mathbb{E}\left(\sup_{k\leq n} |I_k|^2\right) \leq 4\mathbb{E}(|I_n|^2).$$

Hence for $n \leq N$,

$$s(n) \leq 4\mathbb{E}(|I_{T_R \wedge n}|^2) = 4\sum_{k=0}^{n-1} \mathbb{E}\left(|I_{k+1} - I_k|^2 1\{k \leq T_R\}\right). \qquad (2)$$

Then note

$$\begin{aligned} |I_{k+1} - I_k| &= |\widetilde{g}_0(\widetilde{\Gamma}_k(z) - i\theta_{k+1}) - \widetilde{g}_0(\widetilde{\Gamma}_k(z') - i\theta_{k+1})| \\ &\leq \frac{4A\mathbf{c}|\widetilde{\Gamma}_k(z) - \widetilde{\Gamma}_k(z')|}{q(R - k\mathbf{c} - \varepsilon - \delta)} \leq \frac{4A\mathbf{c}(|z - z'| + |I_k|)}{q(R - k\mathbf{c} - \varepsilon - \delta)}. \end{aligned}$$

Proof: bounding $s(N)$

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Proof: bounding $s(N)$ continued

Continuing (**??**),

$$s(n) \leq 4 \sum_{k=0}^{n-1} \mathbb{E}\left(|I_{k+1} - I_k|^2\right) \leq 128 A^2 \mathbf{c}^2 \sum_{k=0}^{n-1} \frac{|z - z'|^2 + s(k)}{q(R - k\mathbf{c} - \varepsilon - \delta)^2}.$$

Grönwall's inequality: we can go from an inequality of the form $x(t) \leq \alpha(t) + \int_0^t \beta(s) x(s) \, \mathrm{d}s \ \forall t \in [0, r]$, to the explicit bound $x(r) \leq \alpha(r) \exp\left(\int_0^r \beta(s) \, \mathrm{d}s\right)$.

By a similar discrete method (and a fiddly calculation), we get $s(N) \leq A' \mathbf{c} |z - z'|^2 / \varepsilon^3$, and so

$$\sup_{k \leq T_R} |M_k(z) - M_k(z')| \leq M |z - z'|^{1/3} \tag{3}$$

for all $z, z' \in \ell_R$, where M is a r.v. with $\mathbb{E}(M^2) \leq A' \mathbf{c} / \varepsilon^3$.

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Proof: at last, our uniform bound

Now by Chebyshev's inequality, picking $L \in \mathbb{N}$

$$\mathbb{P}\left(\sup_{n\le T_R} |M_n(z) - M_n(z')| \ge \varepsilon/2 \text{ for } z, z' \in \ell_R \text{ with } |z - z'| \le \pi/L\right)$$
$$\le \mathbb{P}\left(M \ge \frac{\varepsilon}{2}\left(\frac{L}{\pi}\right)^{1/3}\right) \le \left(\frac{\pi}{L}\right)^{2/3} \frac{A\mathbf{c}}{\varepsilon^5}. \tag{4}$$

Combining this with (**??**), (and using $2\pi i$-periodicity) we get

$$\mathbb{P}\left(\sup_{n\le T_R, z\in\ell_R} \left|\widetilde{\Gamma}_n(z) - (z - n\mathbf{c})\right| \ge \varepsilon\right) \le Le^{-\varepsilon^3/A\mathbf{c}} + \left(\frac{\pi}{L}\right)^{2/3} \frac{A\mathbf{c}}{\varepsilon^5}.$$

Then we get the claimed bound on this probability by choosing an optimal L. $\square$

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$$\leq \mathbb{P}\left(M \geq \frac{\varepsilon}{2}\left(\frac{L}{\pi}\right)^{1/3}\right) \leq \left(\frac{\pi}{L}\right)^{2/3} \frac{A\mathbf{c}}{\varepsilon^5}. \tag{4}$$

Combining this with (**??**), (and using $2\pi i$-periodicity) we get

$$\mathbb{P}\left(\sup_{n \leq T_R, z \in \ell_R} \left|\widetilde{\Gamma}_n(z) - (z - n\mathbf{c})\right| \geq \varepsilon\right) \leq L e^{-\varepsilon^3/A\mathbf{c}} + \left(\frac{\pi}{L}\right)^{2/3} \frac{A\mathbf{c}}{\varepsilon^5}.$$

Then we get the claimed bound on this probability by choosing an optimal L. $\square$

Moral #1: conformal growth is cool

We have seen that the conformal growth setting gives us access to powerful techniques:

- The Carathéodory convergence theorem lets us turn a geometric question about clusters into an analytic question about maps.
- We have explicit estimates for particle maps and their derivatives.
- Harmonic measure can be estimated in terms of the derivative of the cluster map.
- We can change coordinates for convenience much more explicitly than in lattice models.
- For HL(0), there is a *Markov process* associated with the inverse of the cluster map.

Moral #2: other areas are cool too

We also saw lots of useful applications of techniques from other areas of probability and analysis more generally:

- We deal with *harmonic* rather than simply smooth maps so, for example, we can use the maximum principle to bound errors globally using local information.
- We can relate quantities we want to estimate with a martingale evolving as we add more particles.
- We have all the “standard” martingale bounds (Doob’s inequalities, the Azuma-Hoeffding inequality...), and for other models we can use martingale convergence theorems.
- We also often make use of the clever tricks often seen in stochastic analysis (Grönwall’s inequality, Kolmogorov’s lemma, ...).

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