Semimartingale methods for Markov chains, interacting particle systems and random growth models

A series of 8 live-streamed lectures

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LMS PiNE Lectures 2020

PiNE Lectures

Course outline

Foster–Lyapunov methods for Markov chains
Chak Hei Lo (3 lectures on 10–11 September).

Interacting particle systems and martingales
Conrado da Costa (3 lectures on 10–11 September).

Planar random growth and scaling limits
George Liddle and Frankie Higgs (2 lectures on 14 September).

What am I going to cover?

Course outline

Martingale background

Recurrence and transience criteria

Positive recurrence criterion

Example: random walks on half-strips

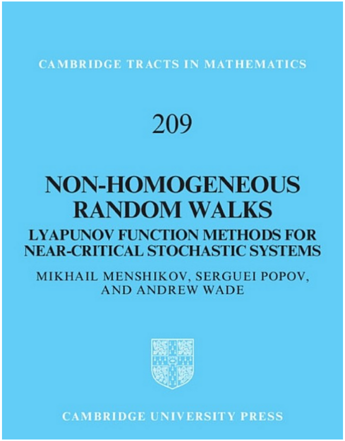
Example: voter model

References and Acknowledgements

Non-homogeneous Random Walks
Cambridge University Press
2016



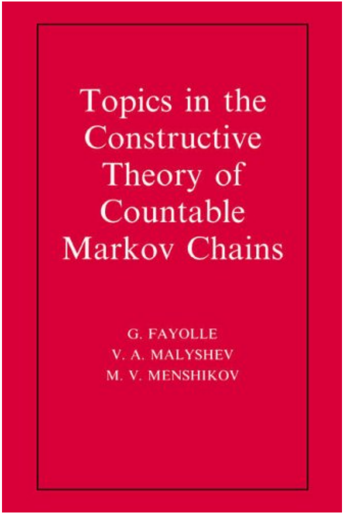
Mikhail Menshikov
Serguei Popov
Andrew Wade



References and Acknowledgements

Topics in the Constructive Theory of Countable Markov Chains
Cambridge University Press
1995

Guy Fayolle
Vadim Malyshev
Mikhail Menshikov



References and Acknowledgements

I am grateful to Andrew Wade for fruitful discussions on the topic of these lectures.

I am thankful to Nicholas Georgiou for the template of the slides.

More references at the end.

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Notation

Suppose that $Z = (Z_n; n \in \mathbb{Z}_+)$ is a real-valued, discrete-time stochastic process adapted to a filtration $(\mathcal{F}_n; n \in \mathbb{Z}_+)$.

The process Z_n is a *martingale* (with respect to the given filtration) if, for all $n \geq 0$,

(i) $\mathbb{E}[|Z_n|] < \infty$, and

(ii) $\mathbb{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] = 0$.

If in (ii) '$=$' is replaced by '$\geq$' (respectively, '$\leq$'), then Z_n is called a *submartingale* (respectively, *supermartingale*).

Martingale background

Theorem 1 (Convergence of non-negative supermartingales)

Suppose $Z_n \geq 0$ is a supermartingale. Then there is an integrable random variable Z such that $Z_n \to Z$ a.s. as $n \to \infty$, and $\mathbb{E}[Z] \leq \mathbb{E}[Z_0]$.

Theorem 2 (Optional stopping for supermartingales)

Suppose $Z_n \geq 0$ is a supermartingale and $\sigma \leq \tau$ are stopping times. Then $\mathbb{E}[Z_\tau] \leq \mathbb{E}[Z_\sigma] < \infty$ and $\mathbb{E}[Z_\tau \mid \mathcal{F}_\sigma] \leq Z_\sigma$ a.s.

Displacement and exit estimates

Theorem 3

Let Z_n be an integrable $\mathcal{F}_n$-adapted process on $\mathbb{R}_+$. Suppose that for some $B \in \mathbb{R}_+$,

$$\mathbb{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] \leq B \quad a.s.$$

Then for any step n and any $x > 0$,

$$\mathbb{P}\left[\max_{0 \leq m \leq n} Z_m \geq z\right] \leq \frac{Bn + \mathbb{E}[Z_0]}{x}.$$

Proof of Theorem 3

Let τ be a stopping time. Then

$$\mathbb{E}\left[Z_{(m+1)\wedge\tau} - Z_{m\wedge\tau} \mid \mathcal{F}_m\right] \le B\,\mathbf{1}\{\tau > m\}.$$

Taking expectations on both sides we get

$$\mathbb{E}\left[Z_{(m+1)\wedge\tau}\right] - \mathbb{E}\left[Z_{m\wedge\tau}\right] \le B\,\mathbb{P}\left(\tau > m\right).$$

Then summing from $m = 0$ to $m = n - 1$ gives

$$\mathbb{E}\left[Z_{n\wedge\tau}\right] - \mathbb{E}\left[Z_0\right] \le B \sum_{m=0}^{n-1} \mathbb{P}\left(\tau > m\right) \le B\,\mathbb{E}[\tau].$$

Displacement and exit estimates

Take $\tau = n \wedge \sigma_x$. Then

$$Bn \geq B\,\mathbb{E}[\tau] \geq \mathbb{E}\left[Z_{n\wedge\sigma_x}\right] - \mathbb{E}\left[Z_0\right].$$

But since $Z_n \geq 0$ we have

$$Z_{n\wedge\sigma_x} \geq x\mathbf{1}\{\sigma_x \leq n\} = x\mathbf{1}\left\{\max_{0\leq m\leq n} Z_m \geq x\right\}$$

and the result follows. $\square$

Example

Let $S_n = \sum_{k=1}^{n} \theta_k$ be simple symmetric random walk on $\mathbb{Z}$. Let $Z_n = S_n^2$. Then

$$\begin{aligned} Z_{n+1} - Z_n = S_{n+1}^2 - S_n^2 &= (S_n + \theta_{n+1})^2 - S_n^2 \\ &= 2S_n\theta_{n+1} + \theta_{n+1}^2. \end{aligned}$$

So

$$\mathbb{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] = 2S_n\mathbb{E}[\theta_{n+1}] + \mathbb{E}[\theta_{n+1}^2] = 1.$$

Hence we have

$$\begin{aligned} \mathbb{P}\left(\max_{0\le m\le n} |S_n| \ge x\right) &= \mathbb{P}\left(\max_{0\le m\le n} Z_m \ge x^2\right) \\ &\le \frac{n}{x^2} \quad \text{for } x > 0. \end{aligned}$$

In this case, Z_n is a submartingale, so one could use the Doob's inequality to get the same result.

Example

Let $u(n) = n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\varepsilon}$ for $\varepsilon > 0$. Then

$$\mathbb{P}\left(\max_{0\leq m\leq n}|S_m| \geq u(n)\right) \leq (\log n)^{-1-2\varepsilon}).$$

Although this seems a rather weak bound, we can still extract a reasonable result by considering the subsequence $n = 2^k$, $k \geq 0$.

Borel-Cantelli shows that $\max_{0\leq m\leq 2^k}|S_m| \leq u(2^k)$ for all but finitely many k, a.s.

Example

Any $n \in \mathbb{N}$ has $2^{k_n} \leq n \leq 2^{k_n+1}$ with $k_n \to \infty$ as $n \to \infty$.

Hence for all but finitely many n,

$$\max_{0 \leq m \leq n} |S_m| \leq \max_{0 \leq m \leq 2^{k_n+1}} |S_m| \leq u(2 \cdot 2^{k_n}) \leq 2u(n).$$

So we have show that for any $\varepsilon > 0$, for all but finitely many n,

$$\max_{0 \leq m \leq n} |S_m| \leq n^{\frac{1}{2}} (\log n)^{\frac{1}{2}+\varepsilon}.$$

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Recurrence classification

Suppose that X_n is an irreducible Markov chain on a countable state space Σ.

- *Recurrent*: With probability 1, for every $x \in \Sigma$, $X_n = x$ infinitely often.
- *Transient*: With probability 1, for every $x \in \Sigma$, $X_n = x$ only finitely often.
- *Positive recurrent*: There exists a probability distribution π on Σ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbf{1}\{X_k = x\} = \pi(x), \quad \text{a.s.},$$

  for all $x \in \Sigma$. Necessarily π is a stationary distribution.
  ($\mathbb{P}(X_n = x) \to \pi(x)$ with some additional aperiodicity.)

Recurrence classification

Equivalent definitions (uses irreducibility and strong Markov heavily):

For a fixed $A \subseteq \Sigma$, we define that $\tau_A = \min\{n \geq 0 : X_n \in A\}$ (stopping / hitting time). We call:

- X_n *recurrent* if for some finite A,

$$\mathbb{P}(\tau_A < \infty \mid X_n = x) = 1 \quad \text{for all } x.$$

- X_n *transient* if for some non-empty A,

$$\mathbb{P}(\tau_A = \infty \mid X_n = x) > 0 \quad \text{for all } x \notin A.$$

- X_n *positive recurrent* if for some finite A,

$$\mathbb{E}[\tau_A \mid X_n = x] < \infty \quad \text{for all } x.$$

Recurrence classification

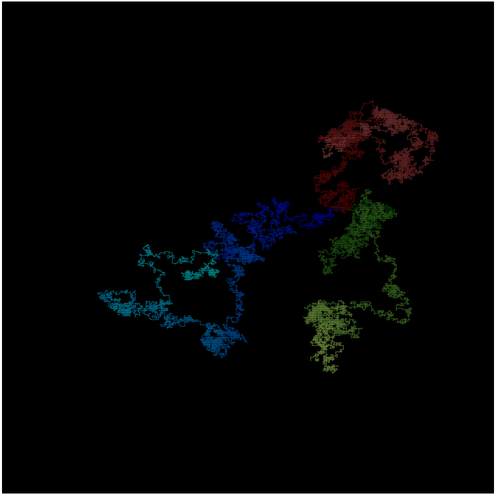
Theorem 4 (Pólya's Recurrence Theorem)

The simple symmetric random walk on $\mathbb{Z}^d$ is recurrent in one or two dimensions, but transient in three or more dimensions.

A quote by Shizuo Kakutani, somewhat 'equivalent' to the theorem.

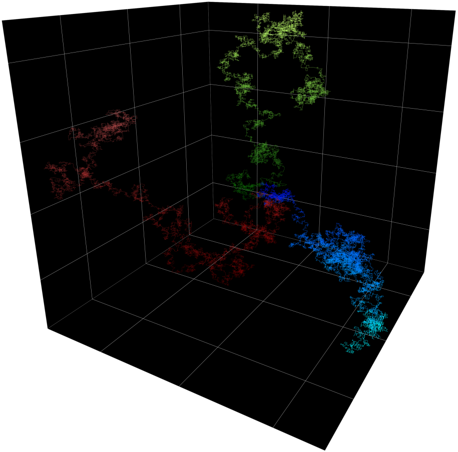
'A drunken man will find his way home,
but a drunken bird may get lost forever.'

Random walk in 2-dimensions



3 simulations on 2-dimensional simple symmetric random walk with 10^5 steps

Random walk in 3-dimensions



3 simulations on 3-dimensional simple symmetric random walk with 10^5 steps

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Recurrence and transience criteria

Theorem 5 (Recurrence criterion)

An irreducible Markov chain X_n on a countably infinite state space Σ is recurrent if and only if there exist a function $f : \Sigma \to \mathbb{R}_+$ and a finite non-empty set $A \subset \Sigma$ such that

$$\mathbb{E}\left[f(X_{n+1}) - f(X_n) \mid X_n = x\right] \leq 0 \quad \textit{for all } x \in \Sigma \backslash A,$$

and $f(x) \to \infty$ as $x \to \infty$.

A weaker version of the 'if' part of this theorem is due to Foster (1953), then improved by Pakes (1969), and the 'only if' part by Mertens et al. (1978).

Example

Let S_n be simple symmetric random walk on $\mathbb{Z}^2$, and consider

$$f(x) = \left(\log(1 + ||x||^2)\right)^{\gamma}$$

for $\gamma \in (0, 1)$. A Taylor's theorem computation gives

$$\mathbb{E}\left[f(S_{n+1}) - f(S_n) \mid S_n = x\right] =$$

$$\gamma(\gamma - 1)||x||^{-2} \left(\log\left(1 + ||x||^2\right)\right)^{\gamma-1} (1 + o(1))$$

which is < 0 for $||x||$ sufficiently large. Hence S_n is recurrent.

Proof of Theorem 5 ('if' part)

Take $X_0 = x \in \Sigma$. Set $Y_n = f(X_{n\wedge\tau_A})$. Then Y_n is a non-negative supermartingale. Hence $Y_n \to Y_\infty$ a.s. for some Y_∞, and

$$\mathbb{E}[Y_\infty \mid X_0 = x] \leq \mathbb{E}[Y_0 \mid X_0 = x] = f(x). \tag{1}$$

On the other hand, since $f \to \infty$, it holds that the set $\{y \in \Sigma : f(y) \leq M\}$ is finite for any $M \in \mathbb{R}_+$, so irreducibility implies that $\limsup_{n\to\infty} f(X_n) = +\infty$ a.s. on $\{\tau_A = \infty\}$.

Hence on $\{\tau_A = \infty\}$ we must have $Y_\infty = \lim_{n\to\infty} Y_n = +\infty$.

This would contradict the inequality (1) if we assume $\mathbb{P}(\tau_A = \infty \mid X_0 = x) > 0$, because then $\mathbb{E}[Y_\infty \mid X_0 = x] = \infty$.

Hence $\mathbb{P}(\tau_A = \infty \mid X_0 = x) = 0$ for all $x \in \Sigma$, which implies recurrence. □

Recurrence and transience criteria

Theorem 6 (Transience criterion)

An irreducible Markov chain X_n on a countably infinite state space Σ is transient if and only if there exist a function $f : \Sigma \to \mathbb{R}_+$ and a non-empty set $A \subset \Sigma$ such that

$$\mathbb{E}\left[f(X_{n+1}) - f(X_n) \mid X_n = x\right] \leq 0 \quad \textit{for all } x \in \Sigma \backslash A,$$

and $f(y) < \inf_{x \in A} f(x)$ for at least one $y \in \Sigma \backslash A$.

A weaker version of this theorem is due to Foster(1953), then improved by Mertens et al. (1978).

Example

Let S_n be simple symmetric random walk on $\mathbb{Z}^d$. Let $\alpha > 0$ and consider the function $f : \mathbb{Z}^d \to (0, 1]$ defined by $f(0) = 1$ and $f(x) = ||x||^{-2\alpha}$ for $x \neq 0$.

A Taylor's theorem computation gives

$$\mathbb{E}\left[f(S_{n+1}) - f(S_n) \mid S_n = x\right] = \frac{\alpha}{d}||x||^{-2-2\alpha}\left(2(\alpha+1) - d + o(1)\right)$$

which is < 0 for $||x||$ sufficiently large provided we choose $\alpha \in \left(0, \frac{d-2}{2}\right)$, which we may do for any $d \geq 3$.

Thus the simple symmetric random walk is transient if $d \geq 3$.

Lemma 7

Let X_n be a Markov chain on state space Σ. Suppose $f : \Sigma \to \mathbb{R}_+$ is measurable, and $A \subseteq \Sigma$ is such that

$$\mathbb{E}\left[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n\right] \le 0 \quad \text{on } \{X_n \in \Sigma \backslash A\}.$$

Then

$$\mathbb{P}(\tau_A < \infty \mid \mathcal{F}_0) \le \frac{f(X_0)}{\inf_{x \in A} f(x)}.$$

Proof of Lemma 7

Set $Y_n = f(X_{n \wedge \tau_A})$. Then Y_n is a non-negative supermartingale, so $Y_n \to Y_\infty$ a.s., and by optional stopping

$$Y_0 \geq \mathbb{E}[Y_\infty \mid \mathcal{F}_0] \geq \mathbb{E}[Y_\infty \mathbf{1}\{\tau_A < \infty\} \mid \mathcal{F}_0].$$

Here

$$\begin{aligned} Y_\infty \mathbf{1}\{\tau_A < \infty\} &= \lim_{n \to \infty} Y_n \mathbf{1}\{\tau_A < \infty\} \\ &= f(X_{\tau_A}) \mathbf{1}\{\tau_A < \infty\} \\ &\geq \inf_{x \in A} f(x) \mathbf{1}\{\tau_A < \infty\} \end{aligned}$$

so

$$f(X_0) = Y_0 \geq \mathbb{P}(\tau_A < \infty) \inf_{x \in A} f(x). \qquad \square$$

Proof of Theorem 6 ('if' part)

With $y \in \Sigma \backslash A$ as stated, the lemma shows that

$$\mathbb{P}(\tau_A < \infty \mid X_0 = y) \le \frac{f(y)}{\inf_{x \in A} f(x)} < 1,$$

which establishes transience. $\square$

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Positive recurrence

Theorem 8

Let $Z_n \in \mathbb{R}_+$ be integrable and $\mathcal{F}_n$-adapted, and let τ be a stopping time. Assume that for some $\varepsilon > 0$,

$$\mathbb{E}\left[Z_{n+1} - Z_n \mid \mathcal{F}_n\right] \le -\varepsilon \quad \text{on } \{\tau > n\}.$$

Then

$$\mathbb{E}[\tau] \le \frac{\mathbb{E}[Z_0]}{\varepsilon} < \infty.$$

Positive recurrence

Proof

We may rewrite the condition as

$$\mathbb{E}\left[Z_{(m+1)\wedge\tau} - Z_{m\wedge\tau} \mid \mathcal{F}_m\right] \le -\varepsilon \mathbf{1}\{\tau > m\}.$$

Taking expectations, we get

$$\mathbb{E}[Z_{(m+1)\wedge\tau}] - \mathbb{E}[Z_{m\wedge\tau}] \le -\varepsilon \mathbb{P}(\tau > m).$$

Hence

$$0 \le \mathbb{E}[Z_{(n+1)\wedge\tau}] \le \mathbb{E}[Z_0] - \varepsilon \sum_{m=0}^{n} \mathbb{P}(\tau > m).$$

Taking $n \to \infty$ we have

$$\mathbb{E}[\tau] = \lim_{n\to\infty} \sum_{m=0}^{n} \mathbb{P}(\tau > m) \le \frac{\mathbb{E}[Z_0]}{\varepsilon} < \infty.$$

$\square$

Positive recurrence

Theorem 9 (Foster's criterion)

An irreducible Markov chain X_n on a countably infinite state space Σ is positive recurrent if and only if there exist a function $f : \Sigma \to \mathbb{R}_+$, a finite non-empty set $A \subset \Sigma$, and $\varepsilon > 0$, such that

$$\begin{aligned} \mathbb{E}\left[f(X_{n+1}) - f(X_n) \mid X_n = x\right] &\leq -\varepsilon \quad \text{for } x \notin A, \\ \mathbb{E}\left[f(X_{n+1}) \mid X_n = x\right] &< \infty \quad \text{for } x \in A. \end{aligned}$$

A weaker version of this theorem is due to Foster(1953), then improved by Mertens et al. (1978) and Mauldon (1957).

Proof ('if' part)
Apply Theorem 8 to the process $f(X_n)$ with stopping time τ_A. □

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Example

The following example is due to Klein Haneveld and Pittenger (1990).

Let ξ_n, $n \geq 0$ be a Markov chain on $\Sigma \subseteq \mathbb{R}_+^2$ with increments $\theta_n = \xi_{n+1} - \xi_n$. Write $\xi_n = \left(\xi_n^{(1)}, \xi_n^{(2)}\right)$.

Let $\tau = \min\left\{n \geq 0 : \xi_n^{(1)}\xi_n^{(2)} = 0\right\}$, the hitting time of $\Sigma_0 = \left\{(x, y) \in \mathbb{R}_+^2 : xy = 0\right\}$.

Suppose that

$$\mathbb{E}\left[\theta_n \mid \xi_n = x\right] = 0 \quad \text{for } x \in \Sigma \backslash \Sigma_0$$

$$\mathbb{E}\left[||\theta_n||^2 \,\middle|\, \xi_n = x\right] \leq B \quad \text{for } x \in \Sigma \backslash \Sigma_0$$

and

$$\mathbb{E}\left[\left(\xi_{n+1}^{(1)} - \xi_n^{(1)}\right)\left(\xi_{n+1}^{(2)} - \xi_n^{(2)}\right) \,\middle|\, \xi_n = x\right] = \rho \quad \text{for } x \in \Sigma \backslash \Sigma_0$$

for a constant covariance ρ.

Example

Then

$$
\begin{aligned}
&\mathbb{E}\left[\xi_{n+1}^{(1)}\xi_{n+1}^{(2)} - \xi_n^{(1)}\xi_n^{(2)} \,\middle|\, \xi_n = x\right] \\
&= \mathbb{E}\left[\left(\xi_{n+1}^{(1)} - \xi_n^{(1)}\right)\left(\xi_{n+1}^{(2)} - \xi_n^{(2)}\right) \,\middle|\, \xi_n = x\right] \\
&\quad + x^{(1)}\mathbb{E}\left[\xi_{n+1}^{(2)} - \xi_n^{(2)} \,\middle|\, \xi_n = x\right] + x^{(2)}\mathbb{E}\left[\xi_{n+1}^{(1)} - \xi_n^{(1)} \,\middle|\, \xi_n = x\right] \\
&= \rho.
\end{aligned}
$$

So if $\rho < 0$ we may apply Theorem 9 with $X_n = \xi_n^{(1)}\xi_n^{(2)}$ to deduce that $\mathbb{E}[\tau] < \infty$.

Remarkably one can also compute $\mathbb{E}[\tau]$ when it exists.

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Example

Suppose that $\mathbb{E}[\tau] < \infty$.

For $k \in \{1, 2\}$, $\xi^{(k)}_{n\wedge\tau}$ is a non-negative martingale that converges to $\xi^{(k)}_{\tau}$.

The associated quadratic variation process satisfies $\langle \xi^{(k)} \rangle_{n\wedge\tau} \le B(n \wedge \tau)$ so the martingales $\xi^{(k)}_{n\wedge\tau}$ are uniformly bounded in L^2, and hence converge in L^2.

Hence $\xi^{(1)}_{n\wedge\tau}\xi^{(2)}_{n\wedge\tau}$ converges in L^1, and

$$\lim_{n\to\infty} \mathbb{E}\left[\xi^{(1)}_{n\wedge\tau}\xi^{(2)}_{n\wedge\tau}\right] = \mathbb{E}\left[\xi^{(1)}_{\tau}\xi^{(2)}_{\tau}\right] = 0$$

Moreover, $\xi^{(1)}_{n\wedge\tau}\xi^{(2)}_{n\wedge\tau} - \rho(n \wedge \tau)$ is a martingale, so

$$\xi^{(1)}_{0}\xi^{(2)}_{0} = \mathbb{E}\left[\xi^{(1)}_{n\wedge\tau}\xi^{(2)}_{n\wedge\tau}\right] - \rho\mathbb{E}[n \wedge \tau].$$

Example

Taking $n \to \infty$ and using monotone convergence we get

$$\mathbb{E}[\tau] = \lim_{n\to\infty} \mathbb{E}[n \wedge \tau] = \frac{\xi_0^{(1)}\xi_0^{(2)}}{|\rho|} \quad \text{when } \rho < 0.$$

A similar argument shows that the result $\mathbb{E}[\tau] < \infty$ when $\rho < 0$ is sharp: if $\rho \geq 0$ and $\xi_0 \notin \Sigma_0$ then $\mathbb{E}[\tau] = \infty$.

For the purpose of deriving a contradiction, suppose $\mathbb{E}[\tau] < \infty$.

Now $\xi_{n\wedge\tau}^{(1)}\xi_{n\wedge\tau}^{(2)}$ is a submartingale, which converges in L^1 as above. Hence

$$0 = \mathbb{E}\left[\xi_{n\wedge\tau}^{(1)}\xi_{n\wedge\tau}^{(2)}\right] \geq \mathbb{E}\left[\xi_0^{(1)}\xi_0^{(2)}\right] > 0,$$

which is a contradiction.

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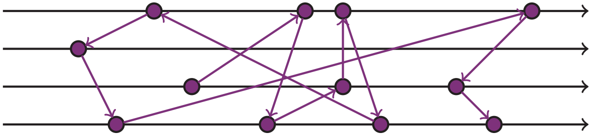
Example: The semi-infinite strip model



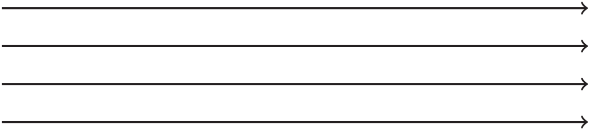
Non-homogeneous Random walk on semi-infinite strip

Let S be a finite non-empty set.

Let Σ be a subset of $\mathbb{R}_+ \times S$ that is *locally finite*, i.e., $\Sigma \cap ([0, r] \times S)$ is finite for all $r \in \mathbb{R}_+$. E.g. $\Sigma = \mathbb{Z}_+ \times S$.

Define for each $k \in S$ the *line* $\Lambda_k := \{x \in \mathbb{R}_+ : (x, k) \in \Sigma\}$. Suppose that for each $k \in S$ the line Λ_k is unbounded.

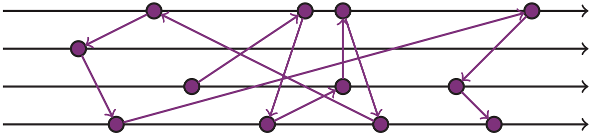
Define the projection of Σ to be $\Lambda := \bigcup_{k \in S} \Lambda_k$.



Non-homogeneous Random walk on semi-infinite strip

Suppose that (X_n, η_n), $n \in \mathbb{Z}_+$, is a time-homogeneous, irreducible Markov chain on Σ, a locally finite subset of $\mathbb{R}_+ \times S$.

Neither coordinate is assumed to be Markov.



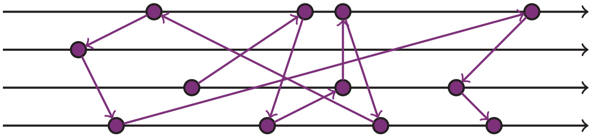
Motivating examples

We can view S as a set of internal states, influencing motion on the lines $\mathbb{R}_+$. E.g.,

Operations research: modulated queues
($S =$ states of server);

Economics: regime-switching processes
(S contains market information);

Physics: physical processes with internal degrees of freedom
($S =$ energy/momentum states of particle).



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Classification of the random walk

Lemma 10

Let (X_n, η_n) be a time-homogeneous irreducible Markov chain on the state-space $\Sigma \in \mathbb{R}_+ \times S$. Exactly one of the following holds:

(i) *If (X_n, η_n) is recurrent, then $\mathbb{P}[X_n = x \ i.o.] = 1$ for any $x \in \Lambda$.*

(ii) *If (X_n, η_n) is transient, then $\mathbb{P}[X_n = x \ i.o.] = 0$ for any $x \in \Lambda$, and $X_n \to \infty$ a.s.*

In the former case, we call (X_n) recurrent, and in the latter case, we call (X_n) transient.

Classification of the random walk

Notice that the process (X_n) is not a Markov chain so this is different from our usual definition.

This is a lemma but not a definition because it is not trivial that the dichotomy of recurrence and transience holds, i.e. the probability must be 0 or 1 rather than other values.

Classification of the random walk

Lemma 11

Let (X_n, η_n) be a time-homogeneous irreducible Markov chain on the state-space Σ. There exists a unique measure $\nu : \Lambda \to \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}\{X_k = x\} = \nu(x), \quad a.s.$$

Exactly one of the following holds.

(i) *If (X_n, η_n) is null, then $\nu(x) = 0$ for all $x \in \Lambda$.*

(ii) *If (X_n, η_n) is positive recurrent, then $\nu(x) > 0$ for all $x \in \Lambda$ and $\sum_{x\in\Lambda} \nu(x) = 1$.*

If X_n is recurrent, then we say that it is null recurrent if (i) holds and positive recurrent if (ii) holds.

Classification of the random walk

This is again a lemma because it is not trivial that the case that $\nu(x) = 0$ for some x and $\nu(x) > 0$ for some other x would not happen.

The proof relies on careful separation of the two coordinates of the state space.

Assumptions

Moments bound on jumps of X_n:

(B_p) $\exists\ C_p < \infty$ s.t.

$$\mathbb{E}[|X_{n+1} - X_n|^p \mid (X_n, \eta_n) = (x, i)] \le C_p.$$

Notation for the expected displacements in the X-coordinate:

$$\mu_i(x) = \mathbb{E}[X_{n+1} - X_n \mid (X_n, \eta_n) = (x, i)].$$

Assumptions (cont.)

Define

$$q_{ij}(x) = \mathbb{P}[\eta_{n+1} = j \mid (X_n, \eta_n) = (x, i)].$$

η_n is "close to being Markov" when X_n is large:

(Q$_\infty$) $q_{ij} = \lim_{x\to\infty} q_{ij}(x)$ exists for all $i, j \in S$ and (q_{ij}) is irreducible.

Let π be the unique stationary distribution on S corresponding to (q_{ij}).

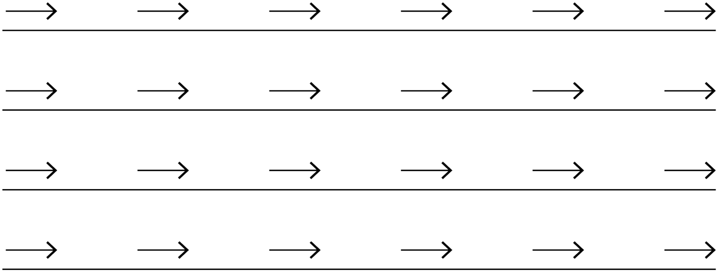
Constant drift

Constant-type drift condition:

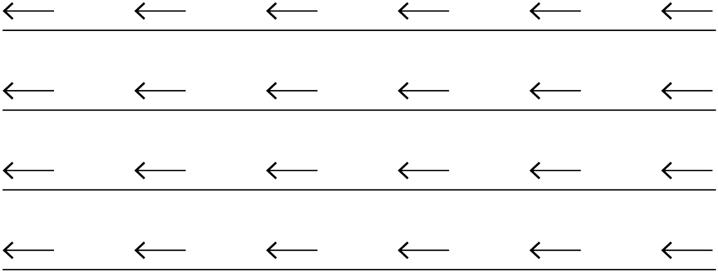
(D_C) $\exists\ d_i \in \mathbb{R}$ for all $i \in S$ such that

$$\mu_i(x) = d_i + o(1).$$

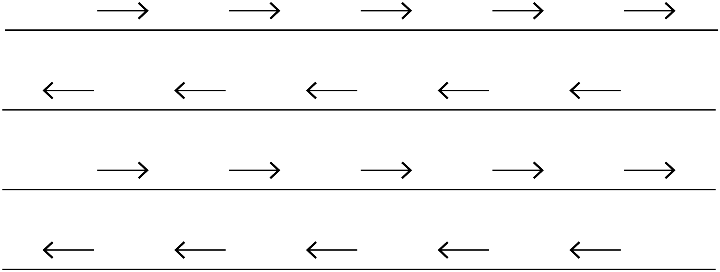
Constant drift



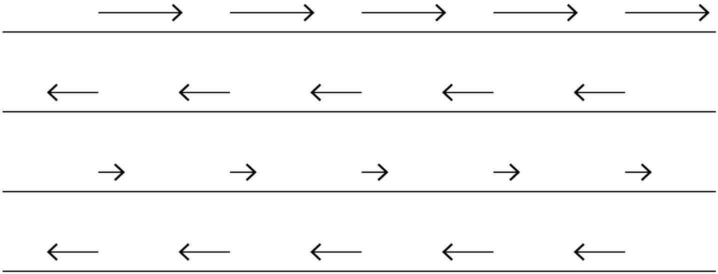
Constant drift



Constant drift



Constant drift



Recurrence classification for constant drift

The following theorem is from Georgiou, Wade (2014), extending slightly earlier work of Malyshev (1972), Falin (1988), and Fayolle et al. (1995).

Theorem 12

Suppose that (B_p) holds for some $p > 1$ and conditions (Q_∞) and (D_C) hold. The following sufficient conditions apply.

- *If* $\sum_{i \in S} d_i \pi_i > 0$*, then* (X_n, η_n) *is transient.*
- *If* $\sum_{i \in S} d_i \pi_i < 0$*, then* (X_n, η_n) *is positive-recurrent.*

Here π_i *is the unique stationary distribution on S.*

The critical case $\sum_{i \in S} d_i \pi_i = 0$ is tortuous and gruelling, but intriguing...

Lyapunov function

Our analysis for the constant drift case is based on various Lyapunov functions. In here I present a choice to prove the positive-recurrent side. Take $g : \Sigma \to (0, \infty)$ where

$$g(x, i) := x + b_i$$

for some $b_i \in \mathbb{R}$.

Increment moment estimates

We will need the following increment moment estimates for our Lyapunov function.

Lemma 13

Suppose that (B_p) *holds for some* $p > 1$ *and conditions* (Q_∞) *and* (D_C) *hold. Then we have, as* $x \to \infty$,

$$\mathbb{E}\left[g(X_1, \eta_1) - g(X_0, \eta_0) \mid (X_0, \eta_0) = (x, i)\right] \\ = d_i + \sum_{j \in S} (b_j - b_i) q_{ij} + o(1).$$

Increment moment estimates

Proof of lemma

Using the condition ($\mathrm{D_C}$) that

$$\mathbb{E}\left[X_1 - X_0 \mid (X_0, \eta_0) = (x, i)\right] = d_i + o(1),$$

we get

$$\begin{aligned}
&\mathbb{E}\left[g(X_1, \eta_1) - g(X_0, \eta_0) \mid (X_0, \eta_0) = (x, i)\right] \\
&= \mathbb{E}\left[X_1 - X_0 \mid (X_0, \eta_0) = (x, i)\right] + \mathbb{E}\left[b_{\eta_1} - b_{\eta_0} \mid (X_0, \eta_0) = (x, i)\right] \\
&= [d_i + o(1)] + \sum_{j \in S} q_{ij}(b_j - b_i),
\end{aligned}$$

by applying (Q_∞) in the last step. Hence we have the result as stated. $\square$

Fredholm alternative

The following well-known algebraic result will enable us to construct the correct Lyapunov function in general for various cases.

Lemma 14 (Fredholm alternative)

Given an $|S| \times |S|$ *matrix* A *and a column vector* $\mathbf{b}$*, the equation* $A\mathbf{a} = \mathbf{b}$ *has a solution* $\mathbf{a}$ *if and only if any column vector* $\mathbf{y}$ *for which* $A^\top \mathbf{y} = \mathbf{0}$ *satisfies* $\mathbf{y}^\top \mathbf{b} = 0$*.*

An important observation

Lemma 15

Let $d_i \in \mathbb{R}$ and (q_{ij}) be an irreducible stochastic matrix with stationary distribution π. Then the following statements are equivalent.

- $\sum_{i \in S} d_i \pi_i = 0.$
- *There exists a solution $\mathbf{a} = (a_1, \ldots, a_{|S|})^\top$ that is unique up to translation to the system of equations*

$$d_i + \sum_{j \in S} (a_j - a_i) q_{ij} = 0, \text{ for all } i \in S. \tag{2}$$

Use of Fredholm alternative

A modification of the above argument yields the following statements, with inequalities instead of equality, which will enable us to show that, under appropriate conditions involving π_j, suitable b_i exist to construct the correct Lyapunov function satisfying appropriate supermartingale conditions in various situations.

Use of Fredholm alternative

Lemma 16

Let $u_i \in \mathbb{R}$ for each $i \in S$.

(i) *Suppose $\sum_{i \in S} u_i \pi_i < 0$. Then there exist $(b_i, i \in S)$ such that*

$$u_i + \sum_{j \in S} (b_j - b_i) q_{ij} < 0, \text{ for all } i \in S.$$

(ii) *Suppose $\sum_{i \in S} u_i \pi_i > 0$. Then there exist $(b_i, i \in S)$ such that*

$$u_i + \sum_{j \in S} (b_j - b_i) q_{ij} > 0, \text{ for all } i \in S.$$

Proof of lemma

We prove only part (i); the proof of (ii) is similar.

Suppose that $\sum_{i\in S} u_i\pi_i = -\varepsilon$ for some $\varepsilon > 0$.

Then taking $\varepsilon_i = \frac{\varepsilon}{|S|\pi_i}$ we get $\sum_{i\in S}(u_i + \varepsilon_i)\pi_i = 0$.

An application of Lemma 15 with $d_i = u_i + \varepsilon_i$ shows that there exist b_i such that

$$u_i + \varepsilon_i + \sum_{j\in S}(b_j - b_i)q_{ij} = 0, \text{ for all } i \in S,$$

which gives the result since $\varepsilon_i > 0$. $\square$

Proof of Theorem

Proof of Theorem 12 ('Positive recurrence side' only)

We will use the Lyapunov function $g(x, i)$ with suitably chosen b_i. First we see that $g(x, i) \to \infty$ as $x \to \infty$. Thus Foster's criterion shows that the process is positive recurrent if

$$\mathbb{E}\left[g(X_{n+1}, \eta_{n+1}) - g(X_n, \eta_n) \mid (X_n, \eta_n) = (x, i)\right] < -\varepsilon \qquad (3)$$

for all sufficiently large x. Now suppose $\sum_{i \in S} d_i \pi_i < 0$, then we use Lemma 16 (i) from our Fredholm alternative corollaries, with $u_i = d_i$ to show that we may choose b_i so that

$$d_i + \sum_{j \in S} (b_j - b_i) q_{ij} < 0.$$

Hence from Lemma 13 we know the condition (3) is satisfied for x sufficiently large. □

Ideas of the complete proof

For the transience part, we can use the Lyapunov function $h_\nu : \Sigma \to (0, \infty)$, $\nu > 0$, defined by

$$h_\nu(x, i) := \begin{cases} x^{-\nu} - \nu b_i x^{-\nu-1} & \text{if } x \geq x_0, \\ x_0^{-\nu} - \nu b_i x_0^{-\nu-1} & \text{if } x < x_0, \end{cases}$$

where $b_i \in \mathbb{R}$ and $x_0 := 1 + 2\nu \max_{i \in S} |b_i|$.

Different drifts

What about $\sum_{i \in S} d_i \pi_i = 0$?

(i) $\sum_{i \in S} d_i \pi_i \neq 0$, constant drift $(\mathrm{D_C})$:

$$\mu_i(x) = d_i + o(1).$$

(ii) $\sum_{i \in S} d_i \pi_i = 0$ and $d_i = 0$ for all i, Lamperti drift $(\mathrm{D_L})$:

$$\mu_i(x) = \frac{c_i}{x} + o(x^{-1})$$

$$\sigma_i(x) = s_i^2 + o(1)$$

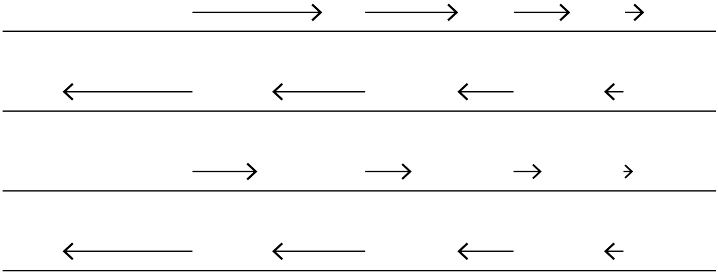
where $\sigma_i(x) = \mathbb{E}[(X_{n+1} - X_n)^2 \mid (X_n, \eta_n) = (x, i)]$.

(iii) $\sum_{i \in S} d_i \pi_i = 0$ and $d_i \neq 0$ for some i, generalized Lamperti drift $(\mathrm{D_G})$:

$$\mu_i(x) = d_i + \frac{e_i}{x} + o(x^{-1})$$

$$\sigma_i(x) = t_i^2 + o(1)$$

Lamperti drift



Lamperti drift

The following theorem is from Georgiou, Wade (2014).

Theorem 17

Suppose that (B_p) holds for some $p > 2$. Suppose also that (Q_∞) and (D_L) hold. Then the following classification applies.

- *If* $\sum_{i \in S}(2c_i - s_i^2)\pi_i > 0$, *then* (X_n, η_n) *is transient.*
- *If* $|\sum_{i \in S} 2c_i\pi_i| < \sum_{i \in S} s_i^2\pi_i$, *then* (X_n, η_n) *is null recurrent.*
- *If* $\sum_{i \in S}(2c_i + s_i^2)\pi_i < 0$, *then* (X_n, η_n) *is positive recurrent.*

If, in addition, (Q_∞^+)and (D_L^+)hold, then the following condition also applies (yielding an exhaustive classification):

- *If* $|\sum_{i \in S} 2c_i\pi_i| = \sum_{i \in S} s_i^2\pi_i$, *then* (X_n, η_n) *is null recurrent.*

More on Lyapunov function

To prove this, we can look at the Lyapunov function $f_\nu : \Sigma \to (0,\infty)$ defined for $\nu \in \mathbb{R}$ by

$$f_\nu(x,i) := \begin{cases} x^\nu + \frac{\nu}{2} b_i x^{\nu-2} & \text{if } x \geq x_0, \\ x_0^\nu + \frac{\nu}{2} b_i x_0^{\nu-2} & \text{if } x < x_0, \end{cases}$$

where $b_i \in \mathbb{R}$ and $x_0 := 1 + \sqrt{|\nu| \max_{i\in S} |b_i|}$.

Generalized Lamperti drift

The following theorem is due to L., Wade (2017).

Theorem 18

Suppose that (B_p) *holds for some* $p > 2$. *Suppose also that* (Q_G) *and* (D_G) *hold. Define* $\mathbf{a} = (a_1, \ldots, a_{|S|})^\top$ *to be a solution to* (2) *whose existence is guaranteed by Lemma 15. Define*

$$U := \sum_{i \in S} \left(2e_i + 2 \sum_{j \in S} a_j \gamma_{ij} \right) \pi_i, \; V := \sum_{i \in S} \left(t_i^2 + 2 \sum_{j \in S} a_j d_{ij} \right) \pi_i.$$

Then the following classification applies.

- *If* $U > V$ *then* (X_n, η_n) *is transient.*
- *If* $|U| < V$ *then* (X_n, η_n) *is null recurrent.*
- *If* $U < -V$ *then* (X_n, η_n) *is positive recurrent.*

Example: One-step Correlated random walk

Suppose that a particle performs a random walk on $\mathbb{Z}_+$ with a short-term memory: the distribution of X_{n+1} depends not only on the current position X_n, but also on the 'direction of travel' $X_n - X_{n-1}$.

Formally, $(X_n, X_n - X_{n-1})$ is a Markov chain on $\mathbb{Z}_+ \times S$ with $S = \{-1, +1\}$, with

$$\mathbb{P}[(X_{n+1}, \eta_{n+1}) = (x + j, j) \mid (X_n, \eta_n) = (x, i)] = q_{ij}(x), \text{ for } i, j \in S.$$

Then for $i \in S$,

$$\mu_i(x) = \mathbb{E}[X_{n+1} - X_n \mid (X_n, \eta_n) = (x, i)] = q_{i,+1}(x) - q_{i,-1}(x).$$

The simplest model has $q_{ii}(x) = q > 1/2$ for $x \geq 1$, so the walker has a tendency to continue in its direction of travel.

Example: One-step Correlated random walk

More generally, suppose that for $q \in (0, 1)$ and constants $c_{-1}, c_{+1} \in \mathbb{R}$ and $\delta > 0$,

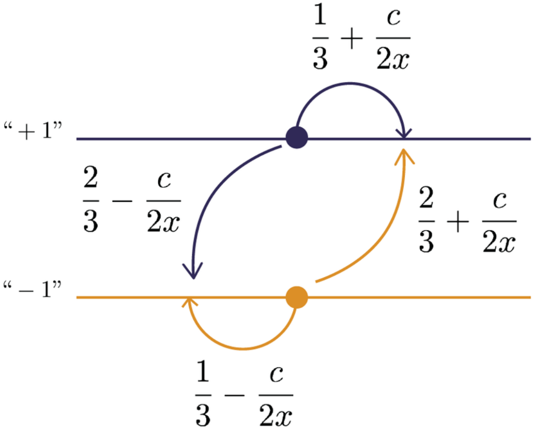
$$q_{ij}(x) = \begin{cases} q + \frac{ic_i}{2x} + O(x^{-1-\delta}) & \text{if } j = i; \\ 1 - q - \frac{ic_i}{2x} + O(x^{-1-\delta}) & \text{if } j \neq i. \end{cases} \tag{4}$$

Here is the recurrence/transience classification for this model.

Theorem 19

Consider the correlated random walk specified by (4)*. Let $c = (c_{+1} + c_{-1})/2$. If $c < -q$, then the walk is positive recurrent. If $c > q$, then the walk is transient. If $|c| \le q$, then the walk is null recurrent.*

Example: One-step Correlated random walk



PiNE Lectures

Outline

Course outline

Martingale background

Recurrence and transience criteria

Positive recurrence criterion

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PiNE Lectures

Voter Model

Consider Markov processes on configurations of particles on $\mathbb{Z}$. So we take as our state space $\{0,1\}^{\mathbb{Z}}$. We call $s \in \{0,1\}^{\mathbb{Z}}$ a *configuration*, and we interpret a coordinate value $s(x) = 1$ as the presence of a particle at the site $x \in \mathbb{Z}$ in the configuration s, and $s(x) = 0$ as the absence of a particle at x.

In our voter model, the dynamics are driven by the presence of discrepancies 01 or 10 in the configuration. In order to obtain well-defined processes, we consider dynamics on configuration with finitely many discrepancies.

At each time step the voter model selects uniformly at random from all discrepancies and then flips the chosen pair to either 00 or 11, with equal chance of each.

Heaviside configuration

Consider the *Heaviside* configuration defined by $\mathbf{1}\{x \le 0\}$, which consists of a single pair 10 abutted by infinite strings of 1s and 0s to the left and right, respectively:

$$\ldots 111111000000 \ldots$$

If the voter model starts from the Heaviside configuration, then at any future time it is a random translate of the same configuration.

Indeed, the position of the rightmost particle performs a symmetric simple random walk. If the voter model starts from a perturbation of the Heaviside configuration, it is natural to study the time it takes to reach a translate of the Heaviside configuration. This example motivates the following notation.

Some notation

Let $\mathcal{S}' \subset \{0,1\}^{\mathbb{Z}}$ denote the set of configurations with a finite number of 0s to the left of the origin and 1s to the right, i.e., $s' \in \{0,1\}^{\mathbb{Z}}$ for which there exist $l, r \in \mathbb{Z}$ with $l < r$ such that $s'(x) = 1$ for all $x \leq l$ and $s'(x) = 0$ for all $x \geq r$.

In other words, $\mathcal{S}'$ contains those configurations of $\{0,1\}^{\mathbb{Z}}$ in which there is only a finite number of discrepancies, and the number of discrepancies of type 10 minus the number of discrepancies of type 01 is equal to 1. Note that $\mathcal{S}'$ is countable.

Let '$\sim$' denote the equivalence relation on $\mathcal{S}'$ such that for $s'_1, s'_2 \in \mathcal{S}'$, $s'_1 \sim s'_2$ if and only if s'_1 and s'_2 are translates of each other, i.e., there exists $y \in \mathbb{Z}$ such that $s'_1(x) = s'_2(x+y)$ for all $x \in \mathbb{Z}$.

Some notation

Now we set $\mathcal{S} := \mathcal{S}'/\sim$. In other words, $\mathcal{S}$ is the set of configurations of the form infinite string of 1s – finite number of 0s and 1s – infinite string of 0s modulo translations.

For example, one $s \in \mathcal{S}$ is

$$s = \ldots 111110101111110000000011010000001000111110000 \ldots$$

Denote $s_H \in \mathcal{S}$ to be the equivalence class of the Heaviside configuration $\mathbf{1}\{x \leq 0\}$.

Voter Model

The *voter model* $(\xi_n, n \geq 0)$ is a time-homogeneous Markov chain on the countable state-space $\mathcal{S}$.

The one-step transition probabilities are determined by the following mechanism.

- At each time step we first choose a discrepancy (i.e. 10 or 01) uniformly at random from the finite number of available discrepancies.
- The chosen pair (10 or 01) will then flips to 00 or 11 each with probability $\frac{1}{2}$.

Some notation

For $s \in \mathcal{S}$, let $N = N(s) \geq 0$ denote the number of 1-blocks not including the infinite 1-block to the left (this is the same as number of 0-blocks not including the infinite 0-block to the right).

Enumerating from left to right, let $n_i = n_i(s)$ denote the size of the i-th 0-block, and $m_i = m_i(s)$ the size of the i-th 1-block.

We can represent configuration $s \in \mathcal{S} \setminus \{s_H\}$ by the vector of block sizes $(n_1, m_1, \ldots, n_N, m_N)$. For example a configuration

$$s = \ldots 11111010111111000000001101000001000111110000\ldots$$

has $N(s) = 6$ and the representation $(1,1,1,6,8,2,1,1,5,1,3,5)$.

Lyapunov Functions

For $s \in \mathcal{S} \backslash \{s_H\}$ and $i \in \{1, 2, \ldots, N\}$ let

$$R_i := R_i(s) := \sum_{j=1}^{i} n_j, \quad \text{and } T_i := T_i(s) := \sum_{j=1}^{i} m_j$$

with the convention that $R_0 = T_{n+1} = 0$.
Define the Lyapunov function $f : \mathcal{S} \to \mathbb{R}_+$ as $f(s_H) = 0$ and for $s \in \mathcal{S} \backslash \{s_H\}$,

$$f(s) := \frac{1}{2} \left(\sum_{i=1}^{N} m_i R_i^2 + \sum_{i=1}^{N} n_i T_i^2 \right).$$

One can check that in fact f is a martingale, i.e.

$$\mathbb{E}\left[f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = s\right] = 0.$$

Recurrence classification

Theorem 20

The voter model is positive recurrent.

This theorem is first proved by Liggett (1976), and the proof that we present here is from Belitsky et al. (2001).

Recurrence classification

Proof

To apply Foster's criterion here, we try to use the function $(f(\xi_n))^\alpha$ for some $\alpha < 1$. From elementary calculus gives us that for $\alpha \in (0,1)$ there exists a positive constant c_1 such that, for all $|x| \le 1$,

$$(1+x)^\alpha - 1 \le \alpha x - c_1 x^2.$$

Using this we evaluate

$$\begin{aligned}
&\mathbb{E}\left[(f(\xi_{n+1}))^\alpha - (f(\xi_n))^\alpha \mid \xi_n = s\right] \\
&= (f(s))^\alpha \mathbb{E}\left[\left(1 + \frac{f(\xi_{n+1}) - f(\xi_n)}{f(\xi_n)}\right)^\alpha - 1 \,\middle|\, \xi_n = s\right] \\
&\le -c_1 (f(s))^{\alpha-2} \mathbb{E}\left[(f(\xi_{n+1}) - f(\xi_n))^2 \,\middle|\, \xi_n = s\right]
\end{aligned}$$

where last step we used the fact that f is a martingale.

Recurrence classification

Define s_0^{-r} be the configuration obtained from s by removing the rightmost 1 for the infinite 1-block and s_N^{+r} be the configuration obtained from s by adding an extra 1 to the right of the Nth 1-block.

We observed that

$$|f(s_0^{-r}) - f(s)| \geq \frac{T_1^2}{2}, \quad \text{and} \quad |f(s_N^{+r}) - f(s)| \geq \frac{R_1^2}{2}.$$

Recurrence classification

It follows from these two inequalities that

$$\begin{aligned}
&\mathbb{E}\left[(f(\xi_{n+1}) - f(\xi_n))^2 \,\middle|\, \xi_n = s\right] \\
&\geq \frac{1}{4N+2}\left(f(s_0^{-r}) - f(s)\right)^2 + \frac{1}{4N+2}\left(f(s_N^{+r}) - f(s)\right)^2 \\
&\geq \frac{T_1^2 + R_N^2}{8N+4} \geq \frac{c_2|s|^4}{N}
\end{aligned}$$

for all $s \in \mathcal{S}$, where $c_2 > 0$ is a constant.

A very important observation here is that the voter model does not increase the number of blocks $N(\xi_n)$. So we have

$$\mathbb{E}\left[(f(\xi_{n+1}) - f(\xi_n))^2 \,\middle|\, \xi_n = s\right] \geq c_0|s|^4, \quad \text{for all } s \in \mathcal{S},$$

with $c_0 = c_2(\xi_0) = \frac{c_2}{N(\xi_0)}$.

Recurrence classification

Putting back we obtain

$$\mathbb{E}\left[(f(\xi_{n+1}))^{\alpha} - (f(\xi_n))^{\alpha} \mid \xi_n = s\right] \leq -c_0 c_1 (f(s))^{\alpha-2}|s|^4.$$

With some work we can also get an upper bound that $f(s) \leq \frac{1}{8}|s|^3$, hence

$$\mathbb{E}\left[(f(\xi_{n+1}))^{\alpha} - (f(\xi_n))^{\alpha} \mid \xi_n = s\right] \leq -8^{\frac{4}{3}} c_0 c_1 (f(s))^{\alpha-\frac{2}{3}}.$$

Take $\alpha = \frac{2}{3}$ to apply the Foster's criterion to complete the proof. □

Remark: By an extension of the Foster argument, using the full range $\alpha < 1$, one can show that for any $\varepsilon > 0$,

$$\mathbb{E}\left[\tau^{\frac{3}{2}-\varepsilon} \,\middle|\, \xi_0 = s\right] < \infty.$$

Exclusion process

Define $p \in [0, 1]$ to be the exclusion parameter. The *exclusion process* $(\xi_n, n \geq 0)$ with the parameter p, denoted as $EP(p)$ is a time-homogeneous Markov chain on the state space $\mathcal{S}$.

At each time step the exclusion process selects uniformly at random from all discrepancies and a chosen pair 01 flips to 10 with probability p (otherwise no move) and a chosen pair 10 flips to 01 with probability $q := 1 - p$ (otherwise no move).

Theorem 21

If $p > \frac{1}{2}$, then $EP(p)$ is positive recurrent. If $p \leq \frac{1}{2}$, then $EP(p)$ is transient.

The positive recurrent side is essentially due to Liggett (1976). The transient side is due to Belitsky et al. (2001).

Exclusion process

Proof (”Positive recurrent” side only)

We will use the same Lyapunov function f as stated before in the voter model.

Let $|s|$ be the length of the string of 0s, and 1s between the infinite string of 1s to the left and the infinite string of 0s to the right.

Since $R_N + T_1 = |s|$, $R_i \geq i$ and $T_i \geq N - i + 1$, we can see that

$$\sum_{i=1}^{N} (R_i + T_i) \geq \max\{|s|, N(N+1)\}.$$

With a calculation, we get

$$\mathbb{E}[f(\xi_{n+1}) - f(\xi_n) | \xi_n = s] \leq \frac{N+1}{2N+1} - (2p-1)\frac{\max\{|s|, N(N+1)\}}{2N+1}.$$

Exclusion process

Suppose $p > \frac{1}{2}$. If $(2p-1)|s| > 2N+2$, we have

$$\mathbb{E}[f(\xi_{n+1}) - f(\xi_n)|\xi_n = s] \leq -\frac{N+1}{2N+1} \leq -\frac{1}{2}.$$

and for $(2p-1)N > 3$, we have

$$\mathbb{E}[f(\xi_{n+1}) - f(\xi_n)|\xi_n = s] \leq 1 - \frac{3(N+1)}{2N+1} \leq -\frac{1}{2}.$$

The set of s for which both $(2p-1)N \leq 3$ and $(2p-1)|s| \leq 2N+2$ is finite, so we have shown that

$$\mathbb{E}[f(\xi_{n+1}) - f(\xi_n)|\xi_n = s] \leq -\frac{1}{2}.$$

for all but finitely many $s \in \mathcal{S}$, and hence by Foster criterion, we proved that $EP(p)$ is positive recurrent when $p > 1/2$.

Hybrid process

Define $\beta \in [0, 1]$ to be the mixing parameter.

The *exclusion-voter process* (or *Hybrid process*) $(\xi_n, n \geq 0)$ with parameters (β, p), denoted as $HP(\beta, p)$ is a time-homogeneous Markov chain on the state space $\mathcal{S}$.

At each time step we decide independently at random whether to perform a *voter* move or an *exclusion* move. We choose a voter move with probability β and an exclusion move with probability $1 - \beta$. Then we execute the chosen move accordingly.

When $\beta = 0$, we recover the exclusion process with parameter p.

When $\beta = 1$, we recover the voter model.

Hybrid process

The following theorems are due to Belitsky et al. (2001).

Theorem 22

If β, p are such that $(1-p)(1-\beta) < \frac{1}{3}$, then $HP(\beta, p)$ is positive recurrent.

Theorem 23

For any $\beta > 0$ and $p \geq \frac{1}{2}$ the process $HP(\beta, p)$ is positive recurrent.

Strength and weakness of the semimartingale methods

Although the methods are very robust and constructive, they are tricky to start with the right Lyapunov functions without any experience.

Without explicit calculation of the expectation, it is hard to tell if the function that we picked is indeed the right one.

The Lyapunov function for a specific model is usually not unique and it can be in various forms.

To pick a good Lyapunov function that enables simpler calculation among all those which will satisfy the conditions in the theorems is a skill derived from experience.

Strength and weakness of the semimartingale methods

A quote from an expert in the semimartingale methods:

'If one cannot use a certain method (other than the semimartingale method) to deduce a recurrence classification, then one can try to use the semimartingale method to deduce the classification.

If one can use a certain method (other than the semimartingale method) to deduce a recurrence classification, then one can surely use the semimartingale method to deduce the same classification.'

References

- **M. Menshikov, S. Popov and A. Wade, *Non-homogeneous Random Walks.* Cambridge University Press, Cambridge, 2016.**
- G. Fayolle, V. A. Malyshev, and M. V. Menshikov, *Topics in the Constructive Theory of Countable Markov Chains*, Cambridge University Press, Cambridge (1995).
- F. G. Foster, On the stochastic matrices associated with certain queuing processes, *Ann. Math. Statistics* **24** (1953) 355–360.
- J.-F. Mertens, E. Samuel-Cahn, and S. Zamir, Necessary and sufficient conditions for recurrence and transience of Markov chains, in terms of inequalities, *J. Appl. Probab.* **15** (1978), no. 4, 848–851.
- A. G. Pakes, Some conditions for ergodicity and recurrence of Markov chains, *Operations Res.* **17** (1969), 1058–1061.
- J. G. Mauldon, On non-dissipative Markov chains, *Proc. Cambridge Philos. Soc.* **53** (1957), 825–835.

References

- L. A. Klein Haneveld and A. O. Pittenger, Escape time for a random walk from an orthant, *Stochastic Process. Appl.* **35** (1990), no. 1, 1–9.
- N. Georgiou and A. R. Wade, Non-homogeneous random walks on a semi-infinite strip, *Stochastic Processes and their Applications* **124** (2014) 3179–3205.
- C. H. Lo and A. R. Wade, Non-homogeneous random walks on a half strip with generalized Lamperti drifts, *Markov Processes and Related Fields* **23** (2017) 125–146.
- C. H. Lo, On some random walk problems, *PhD Thesis*, ArXiV: 1802.06623.
- T. M. Liggett, Coupling the simple exclusion process, *Ann. Probability* **4** (1976), no. 3, 339–356.
- V. Belitsky, P. A. Ferrari, M. V. Menshikov, and S. Yu. Popov, A mixture of the exclusion process and the voter model, *Bernoulli* **7** (2001), no. 1, 119–144.