KNOT THEORY AND QUANTUM GROUPS

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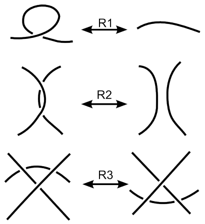
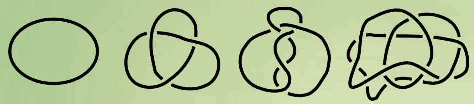
Knots and their Diagrams

A knot is a closed curve in $\mathbb{R}^3$ that does not intersect itself anywhere - like a piece of knotted string, but with the two ends fused together. We may study knots by looking at their projections onto the plane - such a projection is called a knot diagram.



The Reidemeister moves.

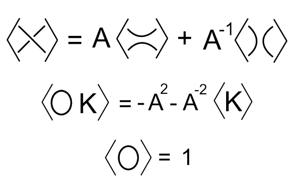
The Theorem of Reidemeister states that two diagrams represent the same knot if and only if one diagram can be changed into the other by a combination of basic topological deformations in the plane, and a finite sequence of *Reidemeister moves*, shown here (where the diagrams represent local changes to a larger knot diagram).



The unknot (left) and three diagrams of the same trefoil knot.

The Bracket Polynomial

In general it is very difficult to find a sequence of Reidemeister moves. Instead, we try to find **knot invariants**, such as the bracket polynomial.
The bracket polynomail is only an invariant of regular isotopy (ignoring the R1 move), but it is very easy to normalise the bracket to obtain an ambient isotopy invariant. After a change of variable this becomes the **Jones Polynomial**; first discovered through work on operator algebras and statistical models.

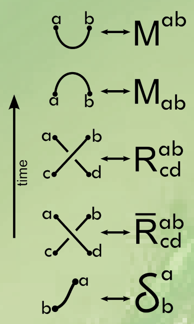


Axioms for the Bracket Polynomial.

The bracket polynomial of a knot K, denoted $\langle K \rangle$, is a Laurnet polynomial in a single variable A, defined by these three axioms.
Where $\langle 0\ K \rangle$ denotes the bracket polynomial of the disjoint union of some knot K and an unknot with no crossings 0.

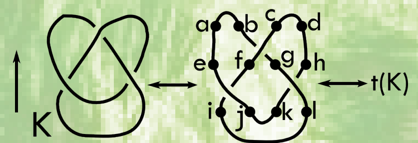
Abstract Tensor Diagrams

We can interpret knot diagrams as diagrammatic representations of matrix multiplication:



By assigning our diagrams a 'time' direction (here going up the page), we may associate matrices to the maxima, minima, and to each of the two types of crossing, as shown. Connected strands thus represent summation over the index corresponding to that line.
This way, any knot diagram is mapped to a specific contracted tensor t(K).

Example: A trefoil knot

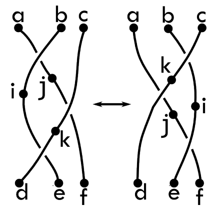


$$t(K) = M_{ab}M_{cd}\delta^a_e\delta^d_h R^{bc}_{fg}\bar{R}^{ef}_{ij}\bar{R}^{gh}_{kl}M^{il}M^{jk}$$

(Using the Einstein summation convention.)

Topological Invariance

By demanding that this particular tensor contraction $t(K)$ is invariant under regular isotopy, we obtain a number of constraints for the given matrices M and R, the most interesting of which is given by the third Reidemeister move:



The third Reidemeister move.

Invariance under this move means that the R matrix must satisfy:

$$R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} = R^{bc}_{ki}R^{ak}_{dj}R^{ji}_{ef}$$

This is in fact the **Yang-Baxter Equation**, which first appeared in the field of statistical mechanics.

Back to the Bracket

If we assume: $M^{ab} = M_{ab}$, then by taking:

$$M = \sqrt{-1}\tilde{\epsilon}, \text{ where } \tilde{\epsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix},$$

$R^{ab}_{cd} = AM^{ab}M_{cd} + A^{-1}\delta^a_c\delta^b_d$ and $\bar{R}^{ab}_{cd} = A^{-1}M^{ab}M_{cd} + A\delta^a_c\delta^b_d$ (All indices run from 1 to 2). It is easy to verify that this satisfies the axioms for the bracket polynomial.
In the special case of the bracket with **A = 1**, we have:
$M = \sqrt{-1}\epsilon$, where $\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
ϵ has the property that for any matrix P with commuting entries; $P\epsilon P^T = DET(P)\epsilon$. Therefore ϵ is the defining invariant for the group $SL(2)$ - the set of matrices with determinant 1.

The Quantum Group $SL(2)_q$

We now ask what structure leaves the deformed epsilon $\tilde{\epsilon}$ invariant? That is, what sort of matrices will satisfy
$(*)\quad P\tilde{\epsilon}P^T = \tilde{\epsilon}$ and $P^T\tilde{\epsilon}P = \tilde{\epsilon}$?
Suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, d are elements of an associative but **non-commutative** ring. If we set $q = \sqrt{A}$, then the equations $(*)$ are equivalent to the set of equations:

$$ba = qab, \quad db = qbd, \quad dc = qcd$$
$$ca = qac, \quad bc = cb, \quad ad - da = (q^{-1} - q)bc$$
$$ad - q^{-1}bc = 1$$

Note that when $q = 1$ these equations once again define $SL(2)$.
If we now define a coproduct Δ, co-unit ϵ, and antipode γ:

$$\Delta(P) = \begin{bmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{bmatrix}, \quad \epsilon(P^i_j) = \delta^i_j,$$

$\gamma(P) = \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}$, then this defines a **Hopf Algebra**.
In fact, this gives a generalisation of $SL(2)$ to a quasi-triangular Hopf algebra, also known as the **Quantum Group**, $SL(2)_q$.

From this there are a number of possible avenues to investigate: How is it that this satisfies the identities required to be quasi-triangular? How and why is $SL(2)_q$ related to a deformation of the Lie Algebra of $SL(2)$?

References
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All diagrams drawn by the author in Inkscape